Singular Homology of non-Archimedean Analytic Spaces and Integration along Cycles

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0 Introduction

0.1 Notation

To begin with, we set a common convention for notation through this paper. We prepare this subsection just for convenience so that it may help a reader to understand this paper well. We do not intend to omit refering to the definitions of new symbols when they appear, and hence there is no need for memorising them. Therefore a reader may skip this subsection if not necessary.

We denote by k the complete non-Archimedean field of characteristic 0 with a multiplicative norm $|\cdot|\colon k \to [0,\infty)$ and by $\tilde{k}$ its residue field of characteristic $p > 0$. Since k has a mixed character, its norm is non-trivial. Throuout this paper, we fix the base field k and an algebraic closure $\overline{k}$ of k. The completion of $\overline{k}$ is C, and the residue field of $\overline{k}$ and C is $\tilde{C}$. In general, for an arbitrary normed ring R, we set $R^\circ \coloneqq \{f \in R \mid \|f\| \leq 1\}$, $R^{\circ\circ} \coloneqq \{f \in R \mid \|f\|_S < 1\} \subset R^\circ$, and $\tilde{R} \coloneqq R^\circ/R^{\circ\circ}$. If we need (finite) subextensions of k (contained in $\overline{k}$), we use K/k and L/k and their residue fields are $\tilde{K}$ and $\tilde{L}$. The absolute Galois groups of $k, K, \ldots$ are $G_k, G_K, \ldots$ The rings of integers of $k, \overline{k}, C, K, \ldots$ are $O_k, O_{\overline{k}}, O_C, O_K, \ldots$ and the maximal ideals of them are $m_k, m_{\overline{k}}, m_C, m_K, \ldots$ When we fix a uniformiser of $k, K, \ldots$ we denote them by $\pi_k, \pi_K, \ldots$ If we assume that k is a local field, i.e. a complete discrete valuation field with a finite residue field $\tilde{k}$, the cardinality of $\tilde{k}, \tilde{K}, \ldots$ are $q_k, q_K, \ldots$ and the absolute ramification indices of $k, K, \ldots$ are $e_k, e_K, \ldots$ See [SER2] for the basic properties of a local field if necessary.

Let K be a (not necessarily complete) non-Archimedean field such as k or $\overline{k}$. When we say a K-algebra, we mean a commutative K-algebra with unit. When we say a normed K-algebra A, we mean a commutative K-algebra with unit and a map $\|\cdot\|$ called a norm satisfying the conditions

$$\|x\| = 0 \text{ if and only if } x = 0 \in A$$
$$\|1\| = 1$$

$$
\begin{aligned}
||f+g|| &\leq \max\{||f||, ||g||\} \\
||fg|| &\leq ||f|| ||g|| \\
||cf|| &= |c| ||f||
\end{aligned}
$$

for any $f, g \in A$ and $c \in K$. For a normed K-algebra A, denote by $\mathscr{M}(A)$ the spectrum ([BER1]) of A, i.e. the set of bounded multiplicative seminorm on A endowed with the weakest topology in which the evaluation map $f \colon \mathscr{M}(A) \to [0, \infty) \colon t \to |f(t)|$ is continuous. If we want to specify the scholar K, we write $\mathscr{M}_K(A)$ instead of $\mathscr{M}(A)$. For a bounded K-homomorphism $\phi \colon A \to B$ between normed K-algebras, denote by $\phi^{\#}$ the continuous map $\mathscr{M}(B) \to \mathscr{M}(A)$ associated with ϕ. For a normed K-algebra A, a normed A-algebra B is a normed K-algebra equiped with a bounded K-homomorphism $A \to B$. Consider the case K is complete. When we say a K-Banach algebra, we mean a normed K-algebra which is complete with respect to its norm. For a K-Banach algebra A, an A-Banach algebra B is a K-Banach algebra with a bounded K-homomorphism $A \to B$. For a normed K-algebra A, denote by $\hat{A}$ or $A^{\wedge}$ the completion of A. A seminormed K-algebra is said to be uniform if its seminorm is power-multiplicative.

We often denote by $A, A', B, \ldots$ arbitrary k-affinoid algebras (see [BER1]), by $V, V', W, \ldots$ arbitrary k-affinoid spaces, and by $A_V, A_{V'}, A_W, \ldots$ the k-affinoid algebras associated with a k-affinoid space $V, W, \ldots$ We write $X, Y, Z, U, \ldots$ for arbitrary k-analytic spaces in the sense of [BER2]. We denote also by $A, A', B, \ldots$ arbitrary k-dagger algebras (see [KLO1]) and X, Y, Z, U arbitrary k-dagger spaces. We do not assume a k-analytic space is strict, good, separated (i.e. Hausdorff), compact, paracompact, connected (= arcwise connected), smooth, or rig-smooth without mentioning. For a k-analytic space X, a k-affinoid algebra A, k-dagger space Y, and k-dagger algebra B, we denote by X_K, A_K, Y_K, and B_K the ground field extensions $X \times_k K$, $A \hat{\otimes}_k K$, $X \times_k K$, and $B \otimes_k^{\dagger} K$. We deal with the class of all isomorphic classes of k-affinoid algebras and k-dagger algebras. Note that the isomorphic class of a k-affinoid algebra is represented by an element of the set $\mathscr{A}_k$ of k-Banach algebras of the form $k\{T_1, \ldots, T_n\}/I$ for an integer $n \in \mathbb{N}$ and a proper ideal $I \subsetneq k\{T_1, \ldots, T_n\}$, which is not a proper class in the sense of Von Neumann-Bernays-Gödel set theory. Therefore we do not have to fix a universe when we want to regard the class of all isomorphic classes of k-affinoid algebras as a set.

Consider the case k is a local field again. We use the following rings:

$$
\begin{aligned}
R &:= \varprojlim_{\mathrm{Frob}} O_{\overline{k}}/pO_{\overline{k}} \cong \varprojlim_{z \mapsto z^p} O_C \\
W(R) &:= \text{the ring of Witt-vectors of } R \\
W(\tilde{k}) &:= \text{the ring of Witt-vectors of } \tilde{k} \\
W(R)_k &:= W(R) \otimes_{W(\tilde{k})} O_k \\
\mathrm{B}_{\mathrm{dR}}^{+} &:= \varprojlim_{N \to \infty} W(R)_k[p^{-1}]/(\ker \theta_k)^N \\
\mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}} &:= \ker \hat{\theta}_k \subset \mathrm{B}_{\mathrm{dR}}^{+}
\end{aligned}
$$

$$\mathrm{B}_{\mathrm{dR}} \quad := \quad \text{the field of fraction of } \mathrm{B}^+_{\mathrm{dR}} = \mathrm{B}^+_{\mathrm{dR}}[\xi^{-1}],$$

where $\theta_k \colon W(R)_k[p^{-1}] \twoheadrightarrow C$ is the canonical k-algebra homomorphism induced by

$$\begin{array}{rcl} \theta \colon W(R) & \twoheadrightarrow & O_C \\ \sum [a_n] p^n & \mapsto & \sum a_n^{(0)} p^n, \end{array}$$

$\hat{\theta}_k \colon \mathrm{B}^+_{\mathrm{dR}} \twoheadrightarrow C$ is the unique continuous extension of $\theta_k \colon W(R)_k[p^{-1}] \twoheadrightarrow C$, and $\xi \in \mathrm{B}^+_{\mathrm{dR}}$ is any generator of the principal ideal $\mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}}$. Here $[\cdot] \colon R \hookrightarrow W(R)$ is Teichmüller embedding $a \mapsto [a] = (a, 0, 0, \ldots)$. We will explain more precise description of the rings above and the topologies of them in §4.1, and see [FON] for more detail.

When multi-indices appear in a calculation, we often use a common convention as below. Let G be an index semigroup, H a semigroup equiped with a compatible semigroup-action of G, and S an index set. For an arbitrary system $a = (a_s)_{s \in S} \in H^S$ and an arbitrary multi-index $g = (g_s)_{s \in S} \in G^S$, if S is a finite set (or if H is a monoid and $a_s^{g_s} = 1$ for almost all $s \in S$), then we denote by a^g the finite (resp. essentially finite) product $\Pi_{s \in S} (a_s)^{g_s}$. If G and H are Abelian semigroups, then we often use the additive notation $ga := \sum_{s \in S} g_s a_s$ instead of the multiplicative one. The most important example of this convention is the case the index semigroup G is the Abelian group $\mathbb{Z}$, the semigroup H is the Abelian multiplicative free monoid generated by variables $T_1, \ldots, T_n$, the index set S is a finite set $\{1, \ldots, n\} \subset \mathbb{N}_+$, and the multi-index semigroup G^S is the Abelian group $\mathbb{Z}^S = \mathbb{Z}^n$. We denote the product $T_1^{I_1} \cdots T_n^{I_n}$ by T^I for each multi-index $I = (I_1, \ldots, I_n) \in \mathbb{Z}^n$. We make use of a similar convention when we denote a Tate algebra and a Monsky-Washnitzer algebra. For each $r = (r_1, \ldots, r_n) \in (0, \infty)^n$, we denote the basic affinoid algebra $k\{r_1^{-1} T_1, \ldots, r_n^{-1} T_n\}$ and the basic dagger algebra $k\{r_1^{-1} T_1, \ldots, r_n^{-1} T_n\}^\dagger$ by $k\{r^{-1} T\}$ and $k\{r^{-1} T\}^\dagger$ for short. In particular if $r = (1, \ldots, 1)$, then we write $k\{T\}$ and $k\{T\}^\dagger$ instead of $k\{r^{-1} T\}$ and $k\{r^{-1} T\}^\dagger$ for short.

We deal with the following categories:

$$\begin{gathered} \mathrm{Set}, \mathrm{Mon}, \mathrm{Grp}, \mathrm{Top}, (\mathrm{B}_{\mathrm{dR}}\text{-Vec}), (k\text{-Alg}), (k\text{-NAlg}), \\ (k\text{-WBanach}), (k\text{-Banach}), (k\text{-An}), (k\text{-Dg}), (\mathrm{Polytope}) \end{gathered}$$

Set is the category of sets whose morphisms are set-theoretical maps; Mon is the category of monoids whose morphisms are monoid homomorphisms; Grp is the category of groups whose morphisms are group homomorphisms; Top is the category of topological spaces whose morphisms are continuous maps; (B_{dR}-Vec) is the category of B_{dR}-linear spaces whose morphisms are B_{dR}-linear homomorphisms; (k-Alg) is the category of k-algebras whose morphisms are k-algebra homomorphisms; (k-NAlg) is the category of normed k-algebras whose morphisms are bounded k-algebra homomorphims; (k-WBanach) $\subset$ (k-NAlg) is the full subcategory of weakly complete k-algebras, Definition 2.1.8; (k-Banach) $\subset$ (k-NAlg) is the full subcategory of k-Banach algebras; (k-An) is the category of k-analytic spaces; (k-Dg) is the category of k-dagger spaces; and

(Polytopes) is the category of polytopes whose morphisms are integral affine maps, Definition 1.1.16.

0.2 Preface

The aim of this paper is to construct a new theory of singular homology of Berkovich's non-Archimedean analytic space (see [BER2]) and integration of a differential form of arbitrary dimension along a cycle in the sense of the analytic homology as a generalisation of Shnirel'man integral. Shnirel'man integral is the integration of a non-Archimedean function defined by Lev Genrikhovich Shnirel'man in [SHN], and is a very useful one in classical p-adic analysis. To begin with, we introduce the idea of the construction of the analytic homology, and adter then we briefly introduce the notion of Shnirel'man integral here.

The most simple way to define a singular homology of an objenct in an arbitrary category $\mathscr{C}$ is the use of the notion of the simplicial category $\triangle$. The simplicial category is the category of finite ordinals (finite ordered sets of natural numbers) whose morphisms are order-preserving maps. Fixing a contravariant functor $\iota\colon \triangle \to \mathscr{C}$, one obtains a chain complex

$$\mathrm{C}_{\cdot}(X) \coloneqq \left(0 \leftarrow \mathbb{Z}^{\oplus \mathrm{Hom}_{\mathscr{C}}(\iota(\{0\},X))} \leftarrow \mathbb{Z}^{\oplus \mathrm{Hom}_{\mathscr{C}}(\iota(\{0,1\},X))} \leftarrow \mathbb{Z}^{\oplus \mathrm{Hom}_{\mathscr{C}}(\iota(\{0,1,2\},X))} \leftarrow \cdots\right),$$

and it gives the homological functor

$$\mathrm{H}_*(X) \coloneqq \mathrm{H}_*(\mathrm{C}_{\cdot}(X))$$

with respect to the simplicial functor ι. Now consider the case $\mathscr{C}$ is the category (k-An) of Berkovich's k-analytic spaces (see [BER1]). There are two simplicial functors, which have the object $\{0,\ldots,n\}$ correspond to the "standard simplices" $\{T_0+\cdots+T_n = 1\} \subset \mathbb{A}_k^{n+1}$ contained in the affine space or $\{T_0 + \cdots + T_n = 1\} \subset \mathrm{D}_k^{n+1} = \mathscr{M}(k\{T_0,\ldots,T_n\})$ contained in the closed unit polydisc for each $n \in \mathbb{N}$. The simplicial functors, of course, give new singular homologies which deeply reflect the information of the analytic structure of an analytic space, but they lack the Galois action. We want to construct the analytic homology which has a canonical Galois action and a canonical Galois equivariant paring with the étale cohomology, and hence we have to seek another simplicial functor. We first cosider the category (Polytope) of polytopes in the Euclidean space $\mathbb{R}^n$ and a realisation functor from (Polytope) to the category (k-Banach) of k-Banach algebras. The realisation functor is faithful, and it canonically fuctors through the forgetful functor from the category (k-BARep) of k-Banach algebra representations of the abosolute Galois group G_k to (k-Banach). Of course, the category (Polytope) is enriched with the canonical simplicial functor which determines the original singular homology in the category Top of topological spaces, and hence one obtains the canonical simplicial functor on (k-BARep). Though the category (k-An) is not contained in (k-Banach), there is obviously the nontrivial intersection: the category (k-Aff) of k-affinoid algebras, or contravariantly equivalently the category of k-affinoid spaces. We construct the bigger category fully faithfully

containing both of the categories (k-An) and (k-Banach). The category is the amalgamated sum (k-Banach) $\cup_{(k\text{-Aff})}$ (k-An). Thus one has obtained the canonical simplicial functor $\iota\colon \triangle \to$ (k-Banach) $\cup_{(k\text{-Aff})}$ (k-An) which canonically fuctors through the category (k-BARep). Regarding an analytic space as an object of (k-Banach) $\cup_{(k\text{-Aff})}$ (k-An), one acquires the analytic simgular homology endowed with the canonical Galois action. In particular considering the case the coefficient group of the analytic singular homology is $\mathbb{Q}_p$ or another field with a Galois action, one can associate an analytic space with a Galois representation in a canonical and functorial way.

$$
\begin{tikzcd}
\mathrm{Top} & & \\
(\mathrm{Polytope}) \arrow[u] \arrow[r] & (k\text{-BARep}) \arrow[d] & \\
\triangle \arrow[u] \arrow[r] & (k\text{-Banach}) \arrow[r] & (k\text{-Banach}) \cup_{(k\text{-Aff})} (k\text{-An}) \\
 & (k\text{-Aff}) \arrow[u] \arrow[r] & (k\text{-An}) \arrow[u]
\end{tikzcd}
$$

Table 1: Categorical aspect

Now we introduce the notion of Shnirel'man integral. Let C be a complete algebraically closed non-Archimedean field with characteristic 0 and residual characteristic $p > 0$, $\mu_n \subset C^\times$ the subgroup of n-th roots of unity for each $n \in \mathbb{N}$, and $\mu \subset C^\times$ the subgroup $\bigcup_{n \in \mathbb{N}} \mu_n$ of roots of unity. Fix points $a \in C$ and $b \in C^\times$, and take an arbitrary subset $D \subset C$ containing the "circle" $\gamma(a,b) := a + b\mu = \{a + b\xi \mid \xi \in \mu\}$ with a centre at a and radius b. For a set-theoretical map $f\colon D \to C$, Shnirel'man defined the integral of f along $\gamma(a,b)$ by the following infinite sum:

Definition 0.2.1 (Shnirel'man integral)**.**

$$\int_{\gamma(a,b)} f(z)dz := \lim_{\substack{n \to \infty \\ p \nmid n}} \frac{1}{n} \sum_{\xi \in \mu_n} f(a + b\xi) b\xi \in C.$$

Note that this limit does not necessarily converge for an arbitrary map f even if f is a locally analytic function such as a locally constant function, and the integrals on $\gamma(a,b')$ and $\gamma(a,b'')$ differ even if $|b'| = |b''|$. Shnirel'man proved that the integral converges if f is presented by a single convergent Laurent series on $\gamma(a,b)$.

Theorem 0.2.2 (Shnirel'man). *Let $k \subset C$ be an arbitrary closed subfield, and suppose $a \in k$ and $b \in k^{\times}$. If there exists some $g \in k\{|b|^{-1}(T-a), |b|(T-a)^{-1}\}$ such that the restriction of f on $\gamma(a,b)$ coincides with g, then the integral of f along $\gamma(a,b)$ converges to the residue $Res(g,a) \in k$ of g at a. Furthermore the integral induces a bounded k-linear functional*

$$\begin{aligned} \int_{\gamma(a,b)} dz \colon H^0\left(a + A_k^1(|b|,|b|), O_{\mathbb{A}_k^1}\right) &\to k \\ f = \sum_{i=-\infty}^{\infty} f_i (T-a)^i &\mapsto Res(f,a) := f_{-1} \end{aligned}$$

with the operator norm

$$\left\| \int_{\gamma(a,b)} dz \right\| = |b|,$$

where $a + A_k^1(|b|,|b|) \subset \mathbb{A}_k^1$ is the annulus $\mathscr{M}(k\{|b|^{-1}(T-a), |b|(T-a)^{-1}\})$.

This is an obvious analogue of the complex integral of a meromorphic function. The counterpart of a cycle in the complex integral is a non-oriented "circle" $\gamma(a,b) = a + b\mu$ each of whose point is presented as a linear expression of a root of unit. Comparing it with the complex residue formula

$$\int_{\theta=0}^{2\pi} f(z(\theta)) dz(\theta) = \pm 2\pi i \mathrm{Res}(f,a),$$

where $z(\theta) := a + be^{\pm 2\pi i\theta}$ for $a \in \mathbb{C}$ and $b \in \mathbb{C}^{\times}$, it is remarkable that no counterparts of the signature $\pm$ corresponding to the orientation of the cycle and the period $2\pi i$ appear in Shnirel'man integral. The reason of the absence is because the analogue of $\pm 2\pi i \in \mathbb{Z}(1) \subset \mathbb{C}$ with an orientation $\pm 1 \in \mathrm{Aut}_{\mathbb{Z}} \mathbb{Z}(1) = \mathbb{Z}^{\times} \subset \mathbb{C}^{\times}$ does not belong to the base field k or even to C, and therefore the definition of Shnirel'man integral is normalised so that such a constant disappears. See the comparison tableau below:

	over $\mathbb{C}/\mathbb{R}$	over $k/\mathbb{Q}_p$
integrant	$f \in \mathrm{H}^0(a + b\mathrm{S}^1, O_{\mathbb{C}})$	$f \in \mathrm{H}^0\left(a + \mathrm{A}(|b|,|b|), O_{\mathbb{A}_k^1}\right)$
integration route	$\gamma \colon [0,1] \to a + bS^1 \subset \mathbb{C}$ $t \mapsto a + be^{\pm 2\pi i t}$ (oriented path)	$\gamma = a + b\mu \subset C$ (not oriented subset)
integral	$\int_{\gamma} f(z)dz = \pm 2\pi i \mathrm{Res}(f,a)$	$\int_{\gamma} f(z)dz = \mathrm{Res}(f,a)$

Table 2: Comparison with complex integral and Shnirel'man integral

The absence of the counterparts of $\pm 2\pi i$ obstructs the extension of Shnirel'man integral of an analytic function on a subset of the affine line $\mathbb{A}^1_k$ to integration of a differential form on a general analytic space, and hence we have to denormalise Shnirel'man integral so that the constant will appear in the equality of residue theorem. In order to denormalise Shnirel'man integral, it is necessary to redefine the notion of an integration route in Shnirel'man integral. It should possess "orientation" and be realised by a "path" with a "period". In fact, the "period" is $\log \epsilon \in \log \mathbb{Z}_p(1) \cong_{\mathbb{Z}_p[G_k]} \mathbb{Z}_p(1)$ lying in Fontaine's p-adic period ring B_{dR} with an orientation in $\mathrm{Aut}_{\mathbb{Z}_p}\mathbb{Z}_p(1) = \mathbb{Z}_p^\times \subset k^\times$. See the desired correspondence below:

	over $\mathbb{C}/\mathbb{R}$	over $k/\mathbb{Q}_p$
period	$\pm 2\pi i \in \mathbb{Z}(1) \subset \mathbb{C}$	$\log \epsilon \in \mathbb{Z}_p(1) \subset \mathrm{B}_{\mathrm{dR}}$
orientation	$\pm 1 \in \mathrm{Aut}_{\mathbb{Z}}\mathbb{Z}(1) = \mathbb{Z}^\times \subset \mathbb{C}^\times$	$u \in \mathrm{Aut}_{\mathbb{Z}_p}\mathbb{Z}_p(1) = \mathbb{Z}_p^\times \subset k^\times$

Table 3: Desired correspondence

Our generalisation of Shnirel'man integral is obtained by regarding a linear expression of systems of power roots of elements in k as an analytic "path" $[0,1] \to \mathbb{A}^1_k$ and using a period in B_{dR}. For example, a typical closed analytic path is given by a system $\epsilon \colon \mathbb{Q} \to \overline{k}^\times$ of primitive power roots of unity, i.e. ϵ is a group homomorphism such that $\epsilon(1) = 1$ and $\epsilon(n^{-1})$ is a primitive n-th root of unity for any $n \in \mathbb{N}_+$. We deal with it as an "exponential-like" function by its formal law of exponent $\epsilon(t)\epsilon(s) = \epsilon(t+s)$. Then the formal linear combination $a + b\epsilon$ corresponds to a set-theoretical map

$$\begin{array}{rccc} \gamma \colon & [0,1] \cap \mathbb{Q} & \to & C \\ & t & \mapsto & a + b\epsilon(t), \end{array}$$

and this is the counterpart of the path

$$\begin{array}{rccc} \gamma \colon & [0,1] & \to & \mathbb{C} \\ & t & \mapsto & a + be^{\pm 2\pi i t} \end{array}$$

in the complex case.

In the similar way, we will define an analytic path from the standard simplex Δ^n and the cube $[0,1]^n$ to a general analytic space. Recall that Jean-Pierre Serre defined the cubical singular homology of a topological space using continuous maps from $[0,1]^n$ in [SER1], and he proved that the cubical singular homology is functorially isomorphic to the singular homology. The construction of the cubical singular homology is quite algebraic, and we obtain the notion of the "analytic cubical singular homology" using analytic paths from $[0,1]^n$ in the same way.

Definition 0.2.3 (analytic (cubical) singular homology). *Let X be a k-analytic space, and M an Abelian group. Define its analytic singular homology and its analytic cubical singular homology with coefficients in M as below:*

$$H_*^{\triangle}(X, M) := H_* \left(0 \leftarrow M^{\oplus X(k)} \leftarrow M^{\oplus Hom(\Delta^1, X)} \leftarrow M^{\oplus Hom(\Delta^2, X)} \leftarrow \cdots \right)$$
$$\text{and} \quad H_*^{\square}(X, M) := H_* \left(0 \to M^{\oplus X(k)} \leftarrow M^{\oplus Hom([0,1],X) \backslash D([0,1],X)} \leftarrow M^{\oplus Hom([0,1]^2,X) \backslash D([0,1]^2,X)} \leftarrow \cdots \right),$$

where $D([0,1]^n, X) \subset Hom([0,1]^n, X)$ is the subset of "degenerate" analytic paths and the boundary map is given in the natural way.

An analytic function on Δ^n or $[0,1]^n$ is an analytic path $\Delta^n \to \mathbb{A}^1_k$ or $[0,1]^n \to \mathbb{A}^1_k$. We will denote by k_{Δ^n} and $k_{[0,1]^n}$ the rings of analytic functions on Δ^n and $[0,1]^n$ respectively. We will define the integration of an analytic function on $[0,1]$, and extend it to the integration of an analytic function on Δ^n or $[0,1]^n$ directly by Fubini's theorem. In this section, we refer only to the cubical cases.

Definition 0.2.4 (Fubini's theorem). *Take any $f \in k_{[0,1]^n}$. Set*

$$\int_{[0,1]^n} f dt_1 \cdots dt_n := \int_0^1 \left(\cdots \int_0^1 \left(\int_0^1 f(t_1, \ldots, t_n) dt_1 \right) dt_2 \cdots \right) dt_n.$$

In fact, the "overconvergence" condition and the "normaslisation" by a constant term $q_k - 1$ are necessary for the integrability, but we omit it now in the introduction. We will precisely deal with it in §4.2. The formal differential form $dt_1 \wedge \cdots \wedge dt_n$ is the canonical volume form on $[0,1]^n$, i.e. the canonical basis of the free $k_{[0,1]^n}$-module generated by the formal simbol $dt_1 \wedge \cdots \wedge dt_n$ itself. The integration of an analytic function on $[0,1]^n$ induces the integration of a differential n-form on $[0,1]^n$ by the natural way.

Definition 0.2.5. *Let $\omega \in k_{[0,1]^n} dt_1 \wedge \cdots \wedge dt_n$ be a differential form $f dt_1 \wedge \cdots \wedge dt_n$ for some $f \in k_{[0,1]^n}$. Set*

$$\int_{[0,1]^n} \omega := \int_{[0,1]^n} f dt_1 \cdots dt_n.$$

Pulling back a differential i-form $\omega \in \mathrm{H}^0(X, \Omega^i_X)$ on a general analytic space X by an analytic path $\gamma \colon [0,1]^i \to X$, we define the integration of ω along γ as below:

Definition 0.2.6.

$$\int_\gamma \omega := \int_{[0,1]^i} \gamma^* \omega.$$

Considering the case $X = \mathbb{G}_{m,k}$, it is easily seen that this integration is the extension of original Shnirel'man integral. Stokes' theorem for this integration will be straightforwardly verified by the formal algebraic argument.

Proposition 0.2.7 (Stokes' theorem)**.** *Let X be a k-analytic space, $\omega \in H^0(X, \Omega_X^i)$ a differential i-form on X, and $\gamma \colon [0,1]^{i+1} \to X$ an analytic path or more generally a cubical singular simplex, i.e. a formal sum of analytic paths as an element of the $(i+1)$-the term $\mathbb{Z}^{\oplus Hom([0,1]^{i+1}, X)}$ of the singular chain complex. Then one has*

$$\int_\gamma d\omega = \int_{\partial\gamma} \omega.$$

Note that the ordinary singular homology of the underlying topological space of an analytic space possesses little information on the analytic structure of it. The analytic homology makes direct use of the analytic structure of an analytic space in the definition, and hence it reflects full information of the analytic structure. The integration of a differential form satisfies many required properties for analysis such as Cauchy's integral theorem, the residue theorem, Cauchy's integral formula, Cauchy-Goursat integral formula, the fundamental theorem of calculus, Fubini's theorem, and Stokes' theorem. Through the integration, we will see some relation between the analytic homology theory and other homology and cohomology theory such as de Rham cohomology, étale cohomology, and higher Chow groups. The integral often tells us whether a given cycle is a boundary or not by Stokes' theorem. The integral gives us specific periods: the values of the integrals of the canonical 1-form of a Tate curve E along two species of characteric cycles are the periods $\log \underline{\epsilon}$ and $\log \underline{q}$, where $\underline{\epsilon}$ (or $\underline{q}$) is a power root system of unity (resp. an element $q \in k$ with a uniformisation $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k}/q^{\mathbb{Z}} \cong_k E$). It implies that the analytic homology group somehow possesses the information of the "intuitive" existence of holes, while the ordinary singular homology of the underlying topological space does not. The reason we emphasised the word "intuitive" is because the purely topological reality sometimes conflicts with the intuition in the non-Archimedean geometry: for example, the affine torus $\mathbb{G}_{m,k}$ is simply connected and a punctured projective line $\mathbb{P}_k^1 \backslash \{a_1, \ldots, a_m\}$ can be simply connected or infinitely disconnected depending on the types (see [BER1]) of the removed points $\{a_1, \ldots, a_m\}$. Although $\mathbb{G}_{m,k}$ is topologically simply connected, the analytic homology group $\mathrm{H}_1(\mathbb{G}_{m,k})$ surely has a cycle rounding the hole at the origin 0. Also note that what is called the axiom of a homology theory and other required properties hold in the analytic homology: the functoriality, existence of the long exact sequence for a space pair, the dimension axiom, the homotopy invariance, existence of the Mayer-Vietoris exact sequence, the excision axiom, the universal coefficient theorem, the relation with the homotopy set π_0, the compatibility of the group structure of an analytic group, and so on.

0.3 Summary

This paper consists of five sections and an appendix. In the first section, we will define k-Banach algebras k_{Δ^n} and $k_{[0,1]^n}$, which we will call the rings of analytic functions on the standard simplex Δ^n and the cube $[0,1]^n$ respectively, and deal with the spectra Δ_k^n

and $[0,1]^n_k$ associated with k_{Δ^n} and $k_{[0,1]^n}$. The algebras k_{Δ^n} and $k_{[0,1]^n}$ are integral domains endowed with powermultiplicative k-algebra norms $\| \cdot \|$; contain infinitely many exponential-like analytic functions satisfying the desired differential equation and the desired integral equation; have a continuous action of the absolute Galois group G_k; and so on. The algebras $k_{[0,1]}$ has the involution $*\colon k_{[0,1]} \to k_{[0,1]}$. The spectra Δ^n_k and $[0,1]^n_k$ are non-empty, connected, compact Hausdorff spaces; the underlying polytopes Δ^n and $[0,1]^n$ are embeded in Δ^n_k and $[0,1]^n_k$ respectively as closed subspaces in the canonical way; and the Shilov boundary (see [DOU] for complex analysis and [BER1] for p-adic analysis) of Δ^n_k and $[0,1]^n_k$ coincide with the vertexes of the embedded underlying polytopes Δ^n and $[0,1]^n$ respectively. They are filled with many interesting structures. Verifying those properties, we will introduce the notion of morphisms from Δ^n_k and $[0,1]^n_k$ to analytic spaces. Such morphisms will be defined with no use of an atlas and a G-topology, and the definition will be quite formal. To tell the truth, the definition of a morphism will be obtained as a morphism in the amalgamated category of the category (k-An) of k-analytic spaces and the category (k-Banach) of unital commutative k-Banach algebras over the category of k-affinoid algebras, which is trivially regarded as the full subcategory of both. Therefore a morphism from $[0,1]$ to an analytic space reflects not only the information of their topology but also their analytic structure. We will introduce the construction of the amalgamated sum of categories in §1.3, but a reader does not have to mind it because we will write the definition of a morphism explicitly.

In the second section, we will give Δ^n_k and $[0,1]^n_k$ the overconvergent structure similar with dagger spaces introduced by Elmar Grosse-Klönne in [KLO1]. Once we endow Δ^n_k and $[0,1]^n_k$ with the overconvergent structure, one can define a morphism to a dagger space in the totally same way. We will verifies the properties of the overconvergent structure of Δ^n_k and $[0,1]^n_k$ analogous to those of the analytic structure of them. Since the proofs do not differ from the corresponding ones in the analytic case, we will often omit them in order to avoid unnecessary repetitions.

In the third section, we will construct the eight singular homologies. The eight types are distinguished by whether an object is an analytic space or a dagger space, whether singular simplices are cubical or not, and whether geometric points are reflected or not. We call them generically the analytic homologies. We will define them by the singular chain complex with respect to morphisms from $\Delta^\cdot_k$ or $[0,1]^\cdot_k$. We will make use of many accustomed ideas in the topological singular homology theory in the proof of basic analogues, and hence [HAT], [SER1], and [MAS] may help readers to understand well. Remark that the analytification functor from [BER1] gives us the definition of the singular homology of an algebraic variety in the sense of schemes. The analytic homologies satisfies the required properties as homology theeories: the functoriality, existence of the long exact sequence for a space pair, the dimension axiom, the homotopy invariance, existence of the Mayer-Vietoris exact sequence, the excision axiom, the universal coefficient theorem, the relation with the homotopy set π_0, the compatibility of the group

structure of an analytic group, and so on. In particular, the homotopy invariance implies

$$\mathrm{H}_n(\mathbb{A}_k^m, \mathbb{Z}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & (n = 0) \\ 0 & (n > 0) \end{cases}$$

for integers $m, n \in \mathbb{N}$ and the affine space $\mathbb{A}_k^m$. It is significant that the analytic homologies has the canonical Galois actions, and hence the analytic homoloogies with a field valued coefficients are Galois representations.

In the fourth section, we will define the integration of a differential form of arbitrary dimension on a dagger space over a local field along a cycle in the sense of the analytic homologies. The integration determines the B_{dR}-valued pairing between the analytic homology and the vector space of grobal sections of the sheaf of differential forms, where the ring B_{dR} is the period ring introduced by Jean-Marc Fontaine in [FON]. For the construction and basic properties of the period ring, we recommend readers seeing [FON] or some alternatives. Now a Stein space has a good property for the cohomology. In particular, it is well known that the de Rham cohomology of a Stein space is canonically isomorphic to the vector space of global sections of the sheaf of differential forms, and hence we will construct a pairing between the analytic homology and the de Rham cohomology of a Stein space such as the analytification of an algebraic variety. Through the comparison isomorphism in the de Rham conjecture (see [TSU],[JON], and [YAM]), we will see the canonical pairing between the analytic homology and the étale cohomology. The isomorphism of the algebraic singular homology and the étale cohomology in [VSF] will give us the canonical pairing between the analytic homology and the algebraic singular homology. In other words, it is the pairing of an analytic cycle and an algebraic cycle. Our integration has a good relation also with the Shnirel'man integral of an analytic function defined by Shnirel'man in [SHN]. The integral of a volume form along a cycle determines the integral of an overconvergent analytic function along a cycle on the affine line $\mathbb{A}_k^1$ and the affine torus $\mathbb{G}_{m,k}$. The integral coincides with the Shnirel'man integral up to the difference of the period $\log \epsilon$ depending on the orientation of the path on which a function is integrated. In particular similar with the Shnirel'man integral, it is impressive that the residue theorem

$$\frac{1}{\log \epsilon} \int_{\gamma} f(T) dT = \mathrm{Res}(f, 0)$$

and Cauchy's integral formula

$$\frac{1}{\log \epsilon} \int_{\gamma} \frac{f(T)}{T - a} dT = f(a)$$

hold in our integration as we mentioned above. Remark that the Abelian group corresponding to the orientation of an orientable manifold is $\mathbb{Z}^{\times} \cong_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ indecating a surface and a back surface, but the corresponding group of our cycles is $\mathbb{Z}_p^{\times} \subset k^{\times}$, which is much greater than $\mathbb{Z}^{\times} \subset \mathbb{C}^{\times}$.

In the fifth section, we will calculate the analytic homologies of the most basic four examples. One of the four is the punctured affine line $\mathbb{G}_{m,k} = \mathbb{A}^1_k \backslash \{0\}$, which is the analytic space obtained as the analytification of the algebraic group $\mathrm{Spec}(k[T_1, T_1^{-1}])$ corresponding to the unit group $k^\times$. For a prime number $l \neq p \in \mathbb{N}$, there are canonical isomorphisms

$$\mathrm{H}_i(\mathbb{G}_{m,k}, \mathbb{Q}_l) \cong_{\mathbb{Q}_l[G_k]} \begin{cases} \mathbb{Q}_l & (i = 0) \\ \mathbb{Q}_l(1) & (i = 1) \end{cases}$$

of l-adic Galois representations. For the prime number $l = p \in \mathbb{N}$ there are a canonical isomorphism

$$\mathrm{H}_0(\mathbb{G}_{m,k}, \mathbb{Q}_p) \cong_{\mathbb{Q}_p[G_k]} \mathbb{Q}_p$$

and a canonical exact sequence

$$0 \to \mathbb{Q}_p(1) \to \mathrm{H}_1(\mathbb{G}_{m,k}, \mathbb{Q}_p) \to k \to 0$$

of p-adic Galois representations. In particular, the p-adic analytic homology $\mathrm{H}_i(\mathbb{G}_{m,k}, \mathbb{Q}_p)$ is a crystalline representation for $i = 0, 1$. It is significant that the analytic homologies of the algebraic group $k^\times$ have the nontrivial 1-cycles such as a "loop" around the origin $0 \in k$, while the underlying topological space of the analytification $\mathbb{G}_{m,k}$ itself is contractible.

1 Affinoid simplex

Let k be a complete non-Archimedean field of characteristic 0, and $\tilde{k}$ its residue field of characteristic $p > 0$. Fix an algebraic closure $\overline{k}$ of k throughout this paper, and denote by C the completion of $\overline{k}$.

In this section, we introduce the ring k_S of (k-valued) analytic functions over an arbitrary polytope $S \subset \mathbb{R}^n$. The ring k_S will be constructed as the completion of a group algebra $k[(\mathbb{Q}_k^\vee)^n]$ of a $\mathbb{Z}$-module $(\mathbb{Q}_k^\vee)^n$ consisting of k-valued exponential-like maps of n-variables with respect to the Gauss seminorm $\| \cdot \|_S$. The ring k_S is a monstrously big, non-Notherian, and topologically uncountably infinitely generated k-algebra whose ordinary residue ring as a normed ring is a fuge one, but has many good properties. For example, it is an integral domain; its Gauss norm is power-multiplicative; it has another handy residue ring than the ordinary residue ring as a normed ring; it has a more intuitive Gel'fand transform than the oridinary Gel'fand transform of a k-Banach algebra; there is a canonical ways of closed embedding ofthe underlying topological space S in its spectrum $\mathscr{M}(k_S)$; the Grothendieck topology of $\mathscr{M}(k_S)$ induced by the ordinary topology of S satisfies Tate's acyclicity; the Shilov boundary of its spectrum is the set of vertices of S; and so on. By the embedding of the underlying topological space S into the spectrum $\mathscr{M}(k_S)$, and general cubes $[0,1]^n \subset \mathbb{R}^n$ are brought in the world of analytic spaces. In particular, there are the interval $[0,1] \subset \mathbb{R}$, the notion of a "path" from $[0,1]$ to an analytic space, the theory of singular homology and cubical singular homology, and the theory of integration of a differential form along a "path".

1.1 Ring of analytic functions

Definition 1.1.1. *Denote by Grp the category of groups.*

Definition 1.1.2. *Set* $\mathbb{Q}^\vee := \mathrm{Hom}_{Grp}(\mathbb{Q}, \overline{k}^\times)$*. Call an element* $x \in \mathbb{Q}^\vee$ *a system of roots of* $x(1)$ *or simply a character. Define the action of* $\mathbb{Q}$ *on* $\mathbb{Q}^\vee$ *by setting* $x^q(t) := x(qt)$ *for each* $q, t \in \mathbb{Q}$ *and* $x \in \mathbb{Q}^\vee$*. This action gives* $\mathbb{Q}^\vee$ *a structure of a* $\mathbb{Q}$*-linear space. Denote by* $\mathbb{Q}^\vee_k \subset \mathbb{Q}^\vee$ *the* $\mathbb{Z}$*-submodule consisting of characters* $x \in \mathbb{Q}^\vee$ *such that* $x(1) \in k^\times$*. Remark that if k is algebraically closed, then* $\mathbb{Q}^\vee_k = \mathbb{Q}^\vee$ *and hence* $\mathbb{Q}^\vee_k$ *is a* $\mathbb{Q}$*-linear space.*

A character $x \in \mathbb{Q}^\vee$ belongs to $\mathbb{Q}^\vee_k$ if and only if $x(\mathbb{Z}) \subset k^\times$. We formally regard characters $x(t) \in \mathbb{Q}^\vee$ as k-valued analytic functions over the line $\mathbb{R}$, though only rational numbers can be substituted for t and $x(t)$ is not necessarily contained in k if $t \in \mathbb{Q}\backslash\mathbb{Z}$. They have the exponential low $x(s+t) = x(s)x(t)$, and therefore can be regarded as some analogues of the exponential map.

Definition 1.1.3. *For each integer* $n \in \mathbb{N}$*, set*

$$E_{k,n} := \bigoplus_{i=1}^{n} \mathbb{Q}^\vee_k \cong_{\mathbb{Z}} (\mathbb{Q}^\vee_k)^n .$$

For each $i = 1, \ldots, n$ *and* $x \in \mathbb{Q}^\vee_k$*, denote by* $x(t_i)$ *the copy of x in the i-th component of* $E_{k,n}$*. The presentation* $x = y_1(t_1) \cdots y_n(t_n)$ *by a* $y = (y_1, \ldots, y_n) \in (\mathbb{Q}^\vee_k)^n$ *is unique for any* $x \in E_{k,n}$*, and set* $x^{(i)} := y_i$ *for each* $i = 1, \ldots, n$*.*

The $\mathbb{Z}$-module $E_{k,n}$ corresponds to the class of complex analytic functions such as $e^{a_1 t_1 + \cdots + a_n t_n}$ for $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. In fact, in order to construct the comparison isomorphism between analytic singular homology and analytic cubical singular homology, which will be defined by the use of analytic functions in §3, we need the wider class of complex analytic functions such as $e^{F(t_1,\ldots,t_n)}$ for a polynomial $F \in \mathbb{C}[t_1, \ldots, t_n]$ without the constant term. See §3.2 for more precise description.

Definition 1.1.4. *For an* $x \in E_{k,n}$*, set* $x(a) = x(a_1, \ldots, a_n) := x^{(1)}(a_1) \cdots x^{(n)}(a_n) \in \overline{k}^\times$ *for each* $a = (a_1, \ldots, a_n) \in \mathbb{Q}^n$ *and set* $|x(a)| = |x(a_1, \ldots, a_n)| := |x^{(1)}(1)|^{a_1} \cdots |x^{(n)}(1)|^{a_n} \in (0, \infty)$ *for each* $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$*. These notations are compatible when* $a \in \mathbb{Q}^n$*. Write* $x(t_1, \ldots, t_n)$ *or* $x(t_0, \ldots, t_{n-1})$ *instead of x if one emphasises the aspect as a group homomorphism* $x \colon \mathbb{Q}^n \to \overline{k}^\times : a \mapsto x(a)$*.*

Definition 1.1.5. *For an Abelian group M, denote by* $k[M]$ *the k-algebra whose underlying k-linear space is* $k^{\oplus M}$ *endowed with the canonical k-basis* $M = 1 \cdot M \subset k^{\oplus M}$ *and whose addition and multiplication are given as*

$$\left(\sum_{x \in M} f_x x\right) + \left(\sum_{x \in M} g_x x\right) := \sum_{x \in M} (f_x + g_x) x$$

$$\text{and} \quad \left(\sum_{x \in M} f_x x\right)\left(\sum_{x \in M} g_x x\right) := \sum_{x \in M} \left(\sum_{y \in M} f_{xy^{-1}} g_y\right) x.$$

Be careful that the sums in the definition above are essentially finite. For an element $f \in k[M]$, denote by $f_x \in k$ the unique coefficient of the k-linear expansion with respect to the k-basis $M \subset k[M]$ so that

$$f = \sum_{x \in M} f_x x.$$

Of course all but finitely many entries of $(f_x)_{x \in M} \in k^M$ are 0.

Definition 1.1.6. *For an integer $n \in \mathbb{N}$, a character $x \in \mathbb{Q}_k^\vee$, and a $\mathbb{Z}$-linear expansion $a_1 t_1 + \cdots + a_n t_n + b \in \mathbb{Z}[t_1, \ldots, t_n]$, set*

$$x(a_1 t_1 + \cdots + a_n t_n + b) \coloneqq x(b) x^{a_1}(t_1) \cdots x^{a_n}(t_n) \in k[E_{k,n}]$$

This notation is compatible with those of the substitution $a \mapsto x(a)$ and the formal symbols $x(t_1), \ldots, x(t_n)$. If k is algebraically closed, then use the same notation for a $\mathbb{Q}$-linear expansion $a_1 t_1 + \cdots + a_n t_n + b \in \mathbb{Q}[t_1, \ldots, t_n]$.

We define the “Gauss norm” and the ring k_S of analytic functions on S.

Definition 1.1.7. *Let $n \in \mathbb{N}$ be an integer. A subset $S \subset \mathbb{R}^n$ is said to be a polytope if S is a non-empty bounded subset and if there exist some integer $m \in \mathbb{N}$ and $\mathbb{Z}$-linear expansions $a_{i,1} t_1 + \cdots + a_{i,n} t_n + b_i \in \mathbb{Z}[t_1, \ldots, t_n]$ for $i = 1, \ldots, m$ such that*

$$S = \left\{ (s_1, \ldots, s_n) \in \mathbb{R}^n \;\middle|\; a_{i,1} s_1 + \cdots + a_{i,n} s_n + b_1 \geq 0, {}^\forall i = 1, \ldots, m \right\}.$$

In paticular a polytope is a non-empty compact topological space.

Definition 1.1.8. *Let $S \subset \mathbb{R}^n$ be a polytope. For each $x(t_1, \ldots, t_n) \in E_{k,n}$ set*

$$\|x\|'_S \coloneqq \sup \{ |x(s)| \mid s \in S \}.$$

Remark that $0 < \|x\|'_S < \infty$ by the non-emptyness and the boundedness of S and by the continuity of the real-valued function $|x(-)| \colon \mathbb{R}^n \to \mathbb{R} \colon a \mapsto |x(a)|$.

Definition 1.1.9. *Let $n \in \mathbb{N}$ be an integer, and S a polytope. Set*

$$L(S) \coloneqq \left\{ a_1 t_1 + \cdots + a_n t_n + b \in \mathbb{Z}[t_1, \ldots, t_n] \;\middle|\; a_1 s_1 + \cdots + a_n s_n + b = 0, {}^\forall (s_1, \ldots, s_n) \in S \right\}$$

and

$$I(S) \coloneqq \sum_{l \in L(S)} \sum_{x \in E_{k,n}} k[E_{k,n}](x(l(t_1, \ldots, t_n))) - 1) \subset k[E_{k,n}].$$

Then $L(S) \subset \mathbb{Z}[t_1, \ldots, t_n]$ is a $\mathbb{Z}$-submodule contained in $\mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n \oplus \mathbb{Z} \subset \mathbb{Z}[t_1, \ldots, t_n]$, and $I(S) \subset k[E_{k,n}]$ is an ideal. Define the Gauss seminorms $\| \cdot \|_S \colon k[E_{k,n}] \to [0, \infty)$ as below:

$$\begin{array}{rcl} \| \cdot \|_S \colon k[E_{k,n}] & \to & [0, \infty) \\ f & \mapsto & \inf\limits_{g \in I(S)} \max\limits_{x \in E_{k,n}} |f_x + g_x| \|x\|'_S \end{array}$$

Definition 1.1.10. *A polytope $S \subset \mathbb{R}^n$ is said to be thick if $I(S) = 0$.*

Be careful that $J(S)$, $I(S)$, and the notion of "thickness" deeply depend on the embedding $S \hookrightarrow \mathbb{R}^n$. If we compose $S \hookrightarrow \mathbb{R}^n$ with the zero section $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$, then $J(S)$ and $I(S)$ will be bigger and the thickness will be lost.

Lemma 1.1.11. *The Gauss seminorms $\|\cdot\|_S \colon k[E_{k,n}] \to [0,\infty)$ is a seminorm of a k-algebra.*

Proof. To begin with, set

$$\|f\|'_S \coloneqq \max_{x \in E_{k,n}} |f_x| \|x\|'_S$$

for each $f \in k[E_{k,n}]$. We verify that $\|\cdot\|'_S \colon k[E_{k,n}] \to [0,\infty)$ is a norm of a k-algebra. It satisfies the strong triangle inequality. Indeed, take an arbitrary $f, g \in k[E_{k,n}]$. By the strong triangle inequality of the norm of k, one has

$$\begin{aligned} \|f+g\|'_S &= \max_{x \in E_{k,n}} |f_x + g_x| \|x\|'_S \leq \max_{x \in E_{k,n}} (\max\{|f_x|, |g_x|\}) \|x\|'_S \\ &= \max\left\{ \max_{x \in E_{k,n}} |f_x| \|x\|'_S, \max_{x \in E_{k,n}} |g_x| \|x\|'_S \right\} = \max\{\|f\|'_S, \|g\|'_S\} \end{aligned}$$

and

$$\begin{aligned} \|fg\|'_S &= \max_{x \in E_{k,n}} \left| \sum_{y \in E_{k,n}} f_{xy^{-1}} g_y \right| \|x\|'_S \leq \max_{x,y \in E_{k,n}} |f_{xy^{-1}}| |g_y| \|x\|'_S = \max_{x,y \in E_{k,n}} |f_x| |g_y| \|xy\|'_S \\ &= \max_{x,y \in E_{k,n}} |f_x| |g_y| \sup_{s \in S} |xy(s)| = \max_{x,y \in E_{k,n}} |f_x| |g_y| \sup_{s \in S} |x(s)| |y(s)| \\ &\leq \max_{x,y \in E_{k,n}} |f_x| |g_y| \left(\sup_{s \in S} |x(s)| \right) \left(\sup_{s \in S} |y(s)| \right) = \max_{x,y \in E_{k,n}} |f_x| |g_y| \|x\|'_S \|y\|'_S \\ &= \left(\max_{x \in E_{k,n}} |f_x| \|x\|'_S \right) \left(\max_{y \in E_{k,n}} |g_y| \|y\|'_S \right) = \|f\|'_S \|g\|'_S. \end{aligned}$$

Therefore $\|\cdot\|_S \colon k[E_{k,n}] \to [0,\infty)$ is a seminorm of a ring. It is a norm because $\|x\|'_S > 0$ for any $x \in E_{k,n}$. The canonical embedding $k \hookrightarrow k[E_{k,n}]$ is trivially an isomorphism with respect to the norm of k and $\|\cdot\|'_S \colon k[E_{k,n}] \to [0,\infty)$ because $\|1(t)\|'_S = 1$. Thus $\|\cdot\|'_S \colon k[E_{k,n}] \to [0,\infty)$ is a norm of a k-algebra.

Now $\|\cdot\|_S$ is the pull-back of the quotient seminorm of $\|\cdot\|'_S$ with respect to the quotient by the ideal $I(S) \subset k[E_{k,n}]$, and it follows that $\|\cdot\|_S$ is also a seminorm of a k-algbra. Remark that we will prove that the ideal $I(S)$ is closed with respect to $\|\cdot\|'_S$ and hence the quotient seminorm $k[E_{k,n}]/I(S) \to [0,\infty)$ is a norm. □

Definition 1.1.12 (ring of analytic functions)**.** *Denote by k_S the completion of $k[E_{k,n}]$ with respect to the Gauss seminorm $\|\cdot\|_S$. Denote also by $\|\cdot\|_S \colon k_S \to [0,\infty)$ the unique continuous extension of $\|\cdot\|_S \colon k[E_{k,n}] \to [0,\infty)$ through the canonical k-algebra homomorphism $k[E_{k,n}] \to k_S$, and call it the Gauss norm of k_S. Call an element of k_S an analytic function on S, or just say it is defined on S. When there is no ambiguity of S, write $\|\cdot\|_S = \|\cdot\|$ for short.*

In other words, we regard the limit of a uniformly convergent infinite sum of linear combinations of characters as an analytic function over S. In this subsection, we see some basic properties of the ring k_S. Unlike a convergent power series, an analytic function in this sense can not necesarrily be uniquely presented as a formal infinite sum. Though an analytic function can be determied by its coefficients, the presentation is not unique. However, in many cases the presentation is unique, and the ring of analytic functions on any polytope S is isometrically isomorphic to the ring of analytic functions on some polytope such that the presentation is unique.

Proposition 1.1.13. *Assume S is thick. For an analytic function $f \in k_S$, there is the unique presentation*

$$f = \sum_{x \in E_{k,n}} f_x x$$

by the coefficient $(f_x)_{x \in E_{k,n}} \in k^{E_{k,n}}$ and hence one has k-linear injective maps

$$k_S \hookrightarrow k^{E_{k,n}} \colon f \mapsto (f_x)_{x \in E_{k,n}}.$$

In particular the canonical k-algebra homomorphism $k[E_{k,n}] \to k_S$ is injectiove and the Gauss seminorm $\| \cdot \|_S \colon k[E_{k,n}] \to [0, \infty)$ is a norm.

Proof. Since S is thick, the Gauss seminorm $\| \cdot \| \colon k[E_{k,n}] \to [0, \infty)$ coincides with the norm $\| \cdot \|'_S \colon k[E_{k,n}] \to [0, \infty)$ defined in the proof of Lemma 1.1.11. Take an arbitary analytic function $f \in k_S$. By the definition of k_S, there is a convergent sequence $(f_i)_{i \in \mathbb{N}} \in k[E_{k,n}]^{\mathbb{N}}$ converging to f with respect to the Gauss norm $\| \cdot \|_S$. Present

$$f_i = \sum_{x \in E_{k,n}} f_{i,x} x$$

by the unique $(f_{i,x})_{x \in E_{k,n}} \in k^{\oplus E_{k,n}}$ for each $i \in \mathbb{N}$. Take any $x \in E_{k,n}$. Since $(f_i)_{i \in \mathbb{N}} \in k[E_{k,n}]^{\mathbb{N}}$ is convergent, one has

$$\lim_{i,j \to \infty} |f_{i,x} - f_{j,x}| \leq \lim_{i,j \to \infty} \sup_{x' \in E_{k,n}} \frac{|f_{i,x'} - f_{j,x'}| \|x'\|_S}{\|x\|_S} = \lim_{i,j \to \infty} \frac{\|f_i - f_j\|_S}{\|x\|_S} = \frac{\lim_{i,j \to \infty} \|f_i - f_j\|_S}{\|x\|_S} = 0.$$

Therefore $(f_{i,x})_{i \in \mathbb{N}}$ also converges to the unique element $f_x \in k$ by the completeness of k. First we verify that

$$f = \sum_{x \in E_{k,n}} f_x x$$

with respect to the topology of k_S, i.e. one has

$$\lim_{F \in \mathscr{F}} \left\| f - \sum_{x \in F} f_x x \right\|_S = 0,$$

where $\mathscr{F} \subset 2^{E_{k,n}}$ is the net of finite subsets of $E_{k,n}$ with the semiorder given by the inclusion relation $\subset$. Take an arbitrary $\epsilon \in (0, \infty)$. Since $(f_i)_{i \in \mathbb{N}}$ is convergent, there exists some $N \in \mathbb{N}$ such that $\| f_i - f_j \|_S < \epsilon$ for any $i, j > N \in \mathbb{N}$. Let $F \in \mathscr{F}$ be the non-empty finite subset of $1 \in E_{k,n}$ and elements $x \in E_{k,n}$ such that $f_{N,x} \neq 0$. Since F is a finite set and since $(f_{i,x})_{i \in \mathbb{N}}$ converges to f_x, there exists some $N' > N \in \mathbb{N}$ such that $|f_{i,x} - f_x| < \|x\|_S^{-1} \epsilon$ for any $x \in F$ and $i \leq N' \in \mathbb{N}$. By the definition of F, for any $x \in E_{k,n} \backslash F$ and $i \geq N' \in \mathbb{N}$, one has $|f_{i,x}| \|x\|_S \leq \| f_i - f_{N'} \|_S < \epsilon$, and therefore $|f_x| \|x\|_S < \epsilon$. For any $i \leq N' \in \mathbb{N}$ and $F' \supset F \in \mathscr{F}$, one obtains

$$\begin{aligned} \left\| f_i - \sum_{x \in F'} f_x x \right\|_S &= \max \left\{ \max_{x \in E_{k,n} \backslash F'} |f_{i,x}| \|x\|_S, \max_{x \in F' \backslash F} |f_{i,x} - f_x| \|x\|_S, \max_{x \in F} |f_{i,x} - f_x| \|x\|_S \right\} \\ &\leq \max \left\{ \max_{x \in E_{k,n} \backslash F} |f_{i.x}| \|x\|_S, \max_{x \in F' \backslash F} |f_x| \|x\|_S, \max_{x \in F} |f_{i,x} - f_x| \|x\|_S \right\} < \epsilon \end{aligned}$$

and

$$\left\| f - \sum_{x \in F'} f_x x \right\|_S \leq \max \left\{ \| f - f_i \|_S, \left\| f - \sum_{x \in F'} f_x x \right\|_S \right\} < \epsilon.$$

Thus one concludes

$$\sum_{x \in E_{k,n}} f_x x := \lim_{F \in \mathscr{F}} \sum_{x \in F} f_x x = f.$$

Suppose there are two presentations

$$f = \sum_{x \in E_{k,n}} f_x x = \sum_{x \in E_{k,n}} g_x x$$

for $(f_x)_{x \in E_{k,n}}, (g_x)_{x \in E_{k,n}} \in k^{E_{k,n}}$. By the continuity of the addition and the multiplication of k_S, one has

$$\sum_{x \in E_{k,n}} (f_x - g_x) x = 0.$$

If $f_x \neq g_x$ for some $x \in E_{k,n}$, then for any $F \in \mathscr{F}$ containing x, one obtains

$$\left\| \sum_{x \in F} (f_x - g_x) x \right\| \geq |f_x - g_x| \|x\|_S > 0$$

and this conflicts with the fact

$$\lim_{F \in \mathscr{F}} \left\| \sum_{x \in F} (f_x - g_x) x \right\| = 0.$$

Therefore $f_x = g_x$ for any $x \in E_{k,n}$, and the presentation

$$f = \sum_{x \in E_{k,n}} f_x x$$

is unique. $\square$

Corollary 1.1.14. *In the situation above, one has*

$$\|f\|_S = \sup_{x \in E_{k,n}} |f_x| \|x\|_S$$

for any $f \in k_S$ and

$$k_S = \left\{ (f_x)_{x \in E_{k,n}} \in k^{E_{k,n}} \;\middle|\; \lim_{x \in E_{k,n}} |f_x| \|x\|_S = 0 \right\}$$

identifying k_S as the k-linear subspace of $k^{E_{k,n}}$ through the embedding $k_S \hookrightarrow k^{E_{k,n}}$, where the limit is the one in the sense of the following definition.

Definition 1.1.15. *Let Λ be a (not necessarily directed) infinite set, X a topological space, and $x \colon \Lambda \to X \colon \lambda \mapsto x_\lambda$ a map of underlying sets. The map x is said to converge to a point $x_\infty \in X$ if for any open neighbourhood $U \subset X$ of x_∞, there exists a finite subset $F \subset \Lambda$ such that $x_\lambda \in U$ for any $\lambda \in \Lambda \backslash F$. The limit x_∞ in this sense is unique if X is Hausdorff, for example. If x converges to the unique point $x_\infty \in X$, we write $x_\infty = \lim_{\lambda \in \Lambda} x_\lambda$.*

Remark that this notion of convergence is compatible with the limit along the index by the unique contable infinite directed set $\mathbb{N}$, i.e. if $\Lambda = \mathbb{N}$, the limit $\lim_{n \in \mathbb{N}}$ in the sense above coincides with the ordinary limit $\lim_{n \to \infty}$. In general, suppose Λ is an infinite directed set. Distinguish the two limits denoting by $\lim_{\lambda \in \Lambda}$ and $\lim_{\lambda \to \infty}$ respectively the limit in the sense above and the ordinary limit by a net. If $\lim_{\lambda \in \Lambda}$ exists, then $\lim_{\lambda \to \infty}$ exists and coincides with $\lim_{\lambda \in \Lambda}$. On the other hand even if $\lim_{\lambda \to \infty}$ exists, $\lim_{\lambda \in \Lambda}$ does not necessarily exist.

This identification $k_S \subset k^{E_{k,n}}$ is useful. For thickpolytopes $S \subset T \subset \mathbb{R}^n$, one can regard k_T as a k-subalgebra of k_S through the indentifications $k_S, k_T \subset k^{E_{k,n}}$ because $\|\cdot\|_S \leq \|\cdot\|_T$, and this identification $k_T \subset k_S$ is compatible with the canonical embeddings $k[E_{k,n}] \hookrightarrow k_S, k_T$. Be careful that this identification $k_T \subset k_S$ is not with their norms $\|\cdot\|_T$ and $\|\cdot\|_S$ in general, but the embedding $k_T \hookrightarrow k_S$ is a contraction map. Call this embedding $k_T \hookrightarrow k_S$ the restriction map. More generally, a continuous map $S \to T$ in a certain class between two polytopes S and T induces a bounded k-algebra homomorphism $k_T \to k_S$.

Definition 1.1.16. *Let $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ be polytopes. A continuous map $a \colon S \to T$ is said to be an affine map if there exist a matrix $A = (A_{j,i})_{j,i} \in M_{m,n}(\mathbb{Q})$ and a collumn vector $b = {}^t(b_1, \ldots, b_m) \in \mathbb{Q}^m$ such that*

$$a(s_1, \ldots, s_n) = \left(\sum_{i=1}^n A_{1,i} s_i + b_1, \ldots, \sum_{i=1}^n A_{m,i} s_i + b_m \right).$$

Note that such a presentation is unique if and only if S is thick. An affine map is said to be integral if a presentation can be taken so that $A \in M_{m,n}(\mathbb{Z})$ and $b \in \mathbb{Z}^m$.

Proposition 1.1.17. *Let $a\colon S \to T$ be an integral affine map between polytopes $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$. Take a presentation by $A \in M_{m,n}(\mathbb{Z})$ and $b \in \mathbb{Z}^m$ of a. There is the unique k-algebra homomorphism $a^*\colon k[E_{k,m}] \to k[E_{k,n}]$ such that*

$$a^*(x(t_1,\ldots,t_m)) = \prod_{j=1}^m x^{(j)}\left(\sum_{i=1}^n A_{j,i}t_i + b_j\right) = \prod_{j=1}^m x^{(j)}(b_j)\prod_{i=1}^n\left(\prod_{j=1}^m x^{(j)A_{j,i}}\right)(t_i)$$

for any $x \in E_{k,m}$. This maps $I(T)$ to $I(S)$, and induces a bounded k-algebra homomorphism $a^*\colon k_T \to k_S$ which is independent of the choice of the presentation by A and b.

Proof. Take any $x, y \in E_{k,m}$. One has

$$\begin{aligned}
&\prod_{j=1}^m (xy)^{(j)}(b_j)\prod_{i=1}^n\left(\prod_{j=1}^m (xy)^{(j)A_{j,i}}\right)(t_i)\\
= \ &\prod_{j=1}^m x^{(j)}(b_j)y^{(j)}(b_j)\prod_{i=1}^n\left(\prod_{j=1}^m x^{(j)A_{j,i}}y^{(j)A_{j,i}}\right)(t_i)\\
= \ &\left(\prod_{j=1}^m x^{(j)}(b_j)\prod_{i=1}^n\left(\prod_{j=1}^m x^{(j)A_{j,i}}\right)(t_i)\right)\left(\prod_{j=1}^m y^{(j)}(b_j)\prod_{i=1}^n\left(\prod_{j=1}^m y^{(j)A_{j,i}}\right)(t_i)\right),
\end{aligned}$$

and hence the k-linear map

$$\begin{aligned}
a^*\colon k[E_{k,m}] &\to k[E_{k,n}]\\
\sum_{x \in E_{k,m}} f_x x &\mapsto \sum_{x \in E_{k,m}} f_x\left(\prod_{j=1}^m x^{(j)}(b_j)\prod_{i=1}^n\left(\prod_{j=1}^m x^{(j)A_{j,i}}\right)(t_i)\right).
\end{aligned}$$

is a k-algebra homomorphism by the universality of the group algebra.

Take any $a_1t_1 + \cdots + a_mt_m + b \in L(T)$ and $x \in \mathbb{Q}_k^\vee$. By the definition of $L(S)$ and $L(T)$, one has

$$\sum_{j=1}^m a_j\left(\sum_{i=1}^n A_{j,i}t_i + b_j\right) + b \in L(S)$$

and hence

$$\begin{aligned}
a^*(x(a_1t_1 + \cdots + a_mt_m + b) - 1) &= a^*\left(x(b)\prod_{j=1}^m x^{a_j}(t_j)\right) - 1 = x(b)\prod_{j=1}^m x^{a_j}\left(\sum_{i=1}^n A_{j,i}t_i + b_j\right) - 1\\
&= x\left(\sum_{j=1}^m a_j\left(\sum_{i=1}^n A_{j,i}t_i + b_j\right) + b\right) - 1 \in I(S).
\end{aligned}$$

It follows that $a^*(I(T)) \subset I(S)$.

Recall that $\|\cdot\|_S \colon k[E_{k,n}] \to [0,\infty)$ and $\|\cdot\|_T \colon k[E_{k,m}] \to [0,\infty)$ are the quotient seminorms of $\|\cdot\|'_S$ and $\|\cdot\|'_T$ with respect to the ideals $I(S)$ and $I(T)$ respectively. Therefore it suffices to show that $a^* \colon k[E_{k,m}] \to k[E_{k,n}]$ is bounded with respect to the norms $\|\cdot\|'_S$ and $\|\cdot\|'_T$. Since $a(S) \subset T$, for each $x \in E_{k,m}$ one obtains

$$\|a^*(x)\|'_S \leq \|x\|'_T$$

by the definition. Therefore for any $f \in k[E_{k,m}]$, it follows

$$\|a^*(f)\|'_S = \left\| \sum_{x \in E_{k,m}} f_x a^*(x) \right\|'_S \leq \max_{x \in E_{k,m}} |f_x| \|a^*(x)\|'_S \leq \max_{x \in E_{k,m}} |f_x| \|x\|'_T = \|f\|'_T,$$

and $a^* \colon k[E_{k,m}] \to k[E_{k,n}]$ is bounded with respect to the norms $\|\cdot\|'_S$ and $\|\cdot\|'_T$.

We verify that a^* is independent of the choice of the presentation by (A, b). This is the unique continuous extention of the k-algebra homomophism

$$a^* \colon k[E_{k,m}]/I(T) \to k[E_{k,n}]/I(S)$$

induced by $a^* \colon k[E_{k,m}] \to k[E_{k,n}]$, through the canonical embeddings $k[E_{k,n}]/I(S) \hookrightarrow k_S$ and $k[E_{k,m}]/I(T) \hookrightarrow k_T$. Suppose there are two presentations of a by $(A, b), (A', b') \in \mathrm{M}_{m,n}(\mathbb{Z}) \times \mathbb{Z}^m$, and let $a^*_{A,b}, a^*_{A',b'} \colon k[E_{k,m}] \to k[E_{k,n}]$ be the k-algebra homomorphisms associated with the representations. It suffices to verify that they induce the same k-algebra homomorphism $k[E_{k,m}]/I(T) \to k[E_{k,n}]/I(S)$, or in other word, $a^*_{A,b}(x) \equiv a^*_{A',b'}(x) \bmod . I(S)$ for any $x \in E_{k,m}$. Take an arbitrary $(s_1, \ldots, s_n) \in S$. Since (A, b) and (A', b') present the same continuous map $a \colon S \to T$, one has

$$\begin{aligned} \sum_{i=1}^{n} (A_{j,i} - A'_{j,i}) s_i + (b_j - b'_j) &= \left(\sum_{i=1}^{n} A_{j,i} s_i + b_j \right) - \left(\sum_{i=1}^{n} A'_{j,i} s_i + b'_j \right) \\ &= a(s_1, \ldots, s_n)_j - a(s_1, \ldots, s_n)_j = 0 \end{aligned}$$

and it implies

$$\sum_{i=1}^{n} (A_{j,i} - A'_{j,i}) t_i + (b_j - b'_j) \in J(S)$$

for each $j = 1, \ldots, m$. Therefore for any $x \in E_{k,m}$ one obtains

$$\begin{aligned} a^*_{A,b}(x) - a^*_{A',b'}(x) &= \prod_{j=1}^{m} x^{(j)} \left(\sum_{i=1}^{n} A_{j,i} t_i + b_j \right) - \prod_{j=1}^{m} x^{(j)} \left(\sum_{i=1}^{n} A'_{j,i} t_i + b'_j \right) \\ &= \left(\prod_{j=1}^{m} x^{(j)} \left(\sum_{i=1}^{n} A'_{j,i} t_i + b'_j \right) \right) \left(\prod_{j=1}^{m} x^{(j)} \left(\sum_{i=1}^{n} (A_{j,i} - A'_{j,i}) t_i + (b_j - b'_j) \right) - 1 \right) \in I(S), \end{aligned}$$

which was what we wanted. $\square$

Corollary 1.1.18. *The k-algebra homomorphism a^* is a contraction map.*

Corollary 1.1.19. *If k is algebraically closed, then an affine map $a \colon S \to T$ induces the bounded k-algebra homomorphism $a^* \colon k_T \to k_S$ which is a contraction map in the similar way.*

Corollary 1.1.20. *Let $a \colon S \to T$ and $b \colon T \to U$ be integral affine maps between polytopes S, T, and U. Then the composition $b \circ a \colon S \to U$ is again an integral affine map and $a^* \circ b^* = (b \circ a)^* \colon k_U \to k_S$.*

Proof. The presentation of $b \circ a$ is given as the product of the presentations of a and b in the natural sense. The equality is trivial from the definition of the bounded k-homomorphisms associated with the integral affine maps. □

Lemma 1.1.21. *Let $S \subset T \subset \mathbb{R}^n$ be thick polytopes. Then the bounded k-algebra homomorphism $k_T \to k_S$ associated with the inclusion $S \hookrightarrow T$, which is obviously an integral affine map, coincides with the restriction map $k_T \to k_S$. In particular it is injective.*

Proof. Trivial. □

Thus the correspondences $k_{(\cdot)} \colon S \rightsquigarrow k_S$ and $* \colon a \rightsquigarrow a^*$ determines a contravariant functor from the category of polytopes whose morphisms are integral affine maps to the category of k-Banach algebras whose morphisms are bounded k-algebra homomorphism. This functor is not full, and hence sends an isomorphism to an isomorphism in a certain narrow class.

Definition 1.1.22. *An integral affine map (or an affine map) $a \colon S \to T$ between polytopes S and T is said to be isomorphic if its inverse map $a^{-1} \colon T \to S$ is also an integral affine map (resp. an affine map).*

Corollary 1.1.23. *Let $a \colon S \to T$ be an isomorphic integral affine map. Then $a^* \colon k_T \to k_S$ is an isometric isomorphism. If k is algebraically closed, the same holds for an isomorphic affine map.*

Proof. The bounded k-algebra homomorphism a^* is an isomorphism because $a^* \circ (a^{-1})^* = \mathrm{id}^* = \mathrm{id}$ and $(a^{-1})^* \circ a^* = \mathrm{id}^* = \mathrm{id}$. It is an isometry because both of a^* and $(a^{-1})^*$ are contraction maps. □

Corollary 1.1.24. *Embedding $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ by the zero section, one has an isometric isomorphism $k_S \cong_k k_{S \times \{0\}}$ for any polytope $S \subset \mathbb{R}^n$. Identify k_S and $k_{S \times \{0\}}$ in this way.*

Thus the definition of k_S is independent of the choice of n such that S is embedded in $\mathbb{R}^n$. Now recall that we rejected the empty set from the definition of a polytope. It is useful to allow one to deal with the empty set because the category of polytopes needs the intersection and the initial object: the intersection of finitely many polytopes is again a polytope or the empty set and the empty set is initial. This extension is pretty important for the theory of sheaves.

Definition 1.1.25. *Formally set* $k_\emptyset := 0$ *and* $\|\cdot\|_\emptyset := 0 \colon k_\emptyset \to [0,\infty)$. *For each polytope* S*, associate the unique embedding* $\emptyset \hookrightarrow S$ *with the bounded* k*-algebra homomorphism* $0 \colon k_S \to k_\emptyset \colon f \mapsto 0$.

Now we prove that the ring k_S of analytic functions on any polytope $S \subset E_{k,n}$ is isometrically isomorphic to the ring k_T of analytic functions on a thick polytope $T \subset \mathbb{R}^m$. Since the embedding $k_T \hookrightarrow k^{E_{k,m}}$ is injective for such a polytope T, fixing an appropriate topological k-basis B of k_S contained in $E_{k,n}$, one obtains an injective k-libear map $k_S \hookrightarrow k^B$. Note that any bounded k-algebra homomorphism $a^* \colon k_T \to k_S$ associated with an integral affine map $a \colon S \to T$ sends each $x \in E_{k,m} \subset k_T$ to a k-linear subspace of the form $ky \subset k_S$ for some $y \in E_{k,n}$, and hence the image of $E_{k,m} \subset k_T$ by an isomorphism $a^* \colon k_T \to k_S$ naturally determines the topological k-basis of k_S contained in $E_{k,n}$.

Proposition 1.1.26 (thick representative)**.** *Let* $S \subset \mathbb{R}^n$ *be a polytope. Then there exist the unique integer* $m \leq n \in \mathbb{N}$ *and a thick polytope* $T \subset \mathbb{R}^m$ *such that* k_S *is isometrically isomorphic to* k_T *through the isomorphism associated with an isomorphic integral affine map* $S \to T$. *Denote by* $\dim S \in \mathbb{N}$ *the unique integer* m.

Proof. First we verify that if S is thick, then $m = n$. Take an integer $m \in \mathbb{N}$ and a thick polytope $T \subset \mathbb{R}^m$ such that k_S is isometrically isomorphic to k_T through the isomorphism associated with an isomorphic integral affine map $S \to T$. Recall that a polytope is a topological manifold with boundary, and its dimension is an invariant. An isomorphic integral affine map induces an isomorphism of topological manifolds with boundaries, and hence preserves the dimensions. Since S and T are thick, the dimensions of S and T as topological manifolds with boundaries are n and m, and therefore $n = m$.

Secondly we verify that if S is not thick, then there exists some polytope $T \subset \mathbb{R}^{n-1}$ such that k_S is isometrically isomorphic to k_T through the bounded k-algebra homomorphism associated with an isomorphic integral affine map $S \to T$. The condition $I(S) \neq 0$ gurantees that $J(S) \neq 0$, ansd one has $J(S) \cap \mathbb{Z} = 0$. Since $\mathbb{Z}t_1 \oplus \cdots \mathbb{Z}t_n \oplus \mathbb{Z}$ is a finitely generated $\mathbb{Z}$-module, there exist a $\mathbb{Z}$-basis $e_1, \ldots, e_{n+1} \in \mathbb{Z}t_1 \oplus \cdots \mathbb{Z}t_n \oplus \mathbb{Z}$ and integers $c_1, \ldots, c_n \in \mathbb{N}$ such that $c_1 \mid \cdots \mid c_n$, $e_{n+1} = 1$, and $J(S) = \mathbb{Z}c_1e_1 \oplus \cdots \oplus \mathbb{Z}c_{n+1}e_{n+1}$. By the definition of $J(S)$, one has

$$J(S) = (J(S) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap (\mathbb{Z}t_1 \oplus \cdots \mathbb{Z}t_n \oplus \mathbb{Z}),$$

and hence $c_i = 0$ or $c_i = 1$ for any $i = 1, \ldots, n+1$. Take an integer $m \leq n \in \mathbb{N}$ such that $c_1 = \cdots = c_{n-m} = 1$ and $c_{n-m+1} = \cdots = c_{n+1} = 0$. By the definition of a polytope, there exist some $l \in \mathbb{N}$ and $f_1, \ldots, f_l \in \mathbb{Z}t_1 \oplus \cdots \mathbb{Z}t_n \oplus \mathbb{Z}$ such that

$$S = \left\{ s \in \mathbb{R}^n \;\middle|\; f_j(s) \leq 0, {}^\forall j = 1, \ldots, l \right\}.$$

Present $f_j(t_1, \ldots, t_n) = f_{j,1}t_1 + \cdots + f_{j,n}t_n + f_{j,n}$ by the unique coefficients $f_{j,1}, \ldots, f_{j,n+1} \in \mathbb{Z}$ for each $j = 1, \ldots, l$. Since $t_1, \ldots, t_n, 1 \in \mathbb{Z}t_1 \oplus \cdots \mathbb{Z}t_n \oplus \mathbb{Z}$ is a $\mathbb{Z}$-basis, there uniquely exists some matrix $F = (F_{j,i})_{j,i} \in \mathrm{GL}_{n+1}(\mathbb{Z})$ such that $t_i = F_{1,i}e_1 + \cdots + F_{n+1,i}e_{n+1}$ for

each $i = 1, \ldots, n$ and $F_{1,n+1} = \cdots = F_{n,n+1} = 0, F_{n+1,n+1} = 1$. Define a continuous map $\pi \colon \mathbb{R}^m \to \mathbb{R}^n$ as

$$\pi(s_1, \ldots, s_m) \coloneqq \left(\sum_{j=1}^{m+1} F_{n-m+j,1} s_j e_{n-m+j}, \ldots, \sum_{j=1}^{m+1} F_{n-m+j,n} s_j e_{n-m+j} \right).$$

This is presented by the composition of a $\mathbb{R}$-linear map given by a matrix in $\mathrm{M}_{n,m}(\mathbb{Z}) \subset \mathrm{M}_{n,m}(\mathbb{R})$ and a parallel translation given by a collumn vector in $\mathbb{Z}^n \subset \mathbb{R}^n$. Set

$$T = \left\{ s \in \mathbb{R}^m \;\middle|\; f_j(\pi(s)) \leq 0, {}^\forall j = 1, \ldots, l \right\}.$$

Remark that $(f_j \circ \pi)(t_1, \ldots, t_m) \in \mathbb{Z}t_1 \oplus \cdots \mathbb{Z}t_n \oplus \mathbb{Z} \subset \mathrm{C}^0(\mathbb{R}^m, \mathbb{R})$ and hence T is linearly convex. The image $\pi(T) \subset \mathbb{R}^n$ is obviously contained in S because $e_1, \ldots, e_{n-m} \in J(S)$. The restriction $\pi|_T \colon T \to S$ is a homeomorphism whose inverse map is the projection

$$\begin{array}{rccc} (e_{n-m+1}, \ldots, e_n) \colon & S & \to & T \\ & s & \mapsto & (e_{n-m+1}(s), \ldots, e_n(s)), \end{array}$$

and hence T is non-empty and compact. It follows that $T \subset \mathbb{R}^m$ is a polytope, and $(e_{n-m+1}, \ldots, e_{n+1}) \colon S \to T$ is an isomorphic integral affine map. induces the isometric isomorphism $k_T \to k_S$. Take an arbitrary $a_1 t_1 + \cdots + a_m t_m + b \in J(T)$. Pulling back it by $(e_{n-m+1}, \ldots, e_n)$, one has $a_1 e_{n-m+1} + \cdots + a_m e_n + b e_{n+1} \in J(S)$, and it implies $a_1 = \cdots = a_m = b = 0$ by the choice of $e_1, \ldots, e_{n+1}$. It follows that $J(T) = 0$ and $I(T) = 0$.

Thirdly if there are two polytopes $T \subset \mathbb{R}^m$ and $T' \subset \mathbb{R}^{m'}$ such that $m, m' \leq n \in \mathbb{N}$, T and T' and thick, and k_S is isometrically isomorphic to k_T and $k_{T'}$ through the bounded k-algebra homomorphisms associated with isometric integral affine maps $S \to T, T'$. Factoring k_S, k_T and $k_{T'}$ are isometrically isomorphic through the bounded k-algebra homomorphisms associated with an isometric integral affine map $T \to T'$. By the first argument, one knows that $m = m'$, and hence m is unique in the second argument. □

Corollary 1.1.27. *For a polytope $S \subset \mathbb{R}^n$, the ideal $I(S) \subset k[E_{k,n}]$ is closed with respect to the norm $\| \cdot \|'_S$ and the canonical k-algebra homomorphism $k[E_{k,n}]/I(S) \to k_S$ is injective.*

Proof. Take a thick polytope $T \subset \mathbb{R}^m$ and an isomorphic integral affine map $a \colon S \to T$. Then the associated k-algebra homomorphism $a^* \colon k[E_{k,m}] \to k[E_{k,n}]/I(S)$ is an isometric isomorphism with respect to the norm $\| \cdot \|_T = \| \cdot \|'_T \colon k[E_{k,m}] \to [0, \infty)$ and the quotient seminorm $\| \cdot \|_S \colon k[E_{k,n}]/I(S) \to [0, \infty)$ of the Gauss seminorm $\| \cdot \|'_S \colon k[E_{k,n}] \to [0, \infty)$. Therefore the quotient seminorm $\| \cdot \|_S \colon k[E_{k,n}]/I(S) \to [0, \infty)$ is a norm, and it follows the desired facts. □

By the proposition above, we will reduce many definitions and propositions to the case polytopes are thick. Be careful that the such a polytope $T \subset \mathbb{R}^m$ and an isomorphic integral affine map $S \to T$ in the proposition above are not unique, while the integer $m = \dim S \leq n \in \mathbb{N}$ is unique. Therefore there are no canonical isomorphism between k_T

and k_S, and one should not and do not indentify k_S and k_T. We will deal with the standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$, which is "non-canonically" homeomorphic to the polytope

$$\left\{ s \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} s_i \leq 1, 0 \leq s_i, {}^{\forall} i = 1, \ldots, n \right\}.$$

There are the canonical collection of $n+1$ homeomorphisms, though. The reason why one should identify Δ^n with the polytope above through one of the $n+1$ homeomorphism is because this deprive the canonical construction of the analytic singular homology. We will define the analytic singular homology using Δ^n in §3.1.

Now we prepare for some useful operations for the rings of analytic functions. First we prove the compatibility of the Gauss norm with the base change.

Proposition 1.1.28 (ground field extension)**.** *Let S be a polytope, and K/k an extension of complete non-Archimedean fields. The canonical bounded k-algebra homomorphism $k_S \hat{\otimes}_k K \to K_S$ is an isometric isomorphism onto the image.*

Be careful about the fact that $k_S \hat{\otimes}_k K \to K_S$ is not surjective unless $K = k$, because $\mathbb{Q}_k^{\vee} \subsetneqq \mathbb{Q}_K^{\vee}$ for any non-trivial extension K/k.

Proof. Take a thick polytope $T \subset \mathbb{R}^m$ such that k_S is isometrically isomorphic to k_T through the bounded k-algebra homomorphism $a^* \colon k_T \to k_S$ associated with an isomorphic integral affine map $S \to T$. Then the diagram

$$\begin{array}{ccc} k_T \hat{\otimes}_k K & \longrightarrow & K_T \\ \big\downarrow & & \big\downarrow \\ k_S \hat{\otimes}_k K & \longrightarrow & K_S \end{array}$$

commutes by the definition of the bounded k-algebra homomorphism associated with an integral affine map, and hence replacing S to T, we may and do assume that S is thick. Then the identification

$$k_S = \left\{ (f_x)_{x \in E_{k,m}} \in k^{E_{k,m}} \;\middle|\; \lim_{x \in E_{k,m}} |f_x| \|x\|_S = 0 \right\}$$

implies the desired fact. □

As k-vector spaces, one has a natural product $k[E_{k,n}] \otimes_k k[E_{k,m}] \to k[E_{k,n+m}]$ induced by the identification

$$\begin{array}{rcl} \otimes \colon E_{k,n} \times E_{k,m} & \to & E_{k,n+m} \\ (x, y) & \mapsto & x \otimes y \coloneqq x^{(1)}(t_1) \cdots x^{(n)}(t_n) y^{(1)}(t_{n+1}) \cdots y^{(m)}(t_{n+m}). \end{array}$$

We extend this map to k_S's.

Definition 1.1.29 (fibre product). *Let $f_i \in k_{S_i}$ be an analytic function over a thick polytope $S_i \subset \mathbb{R}^{n_i}$ for each $i = 1, \ldots, m$. Identifying $E_{k,n_1} \times \cdots \times E_{k,n_m}$ with $E_{k,n_1+\cdots n_m}$, one obtains the tensor product*

$$\begin{aligned} f_1 \otimes \cdots \otimes f_m &\coloneqq \sum_{(x_1,\ldots,x_m)\in E_{k,n_1}\times\cdots\times E_{k,n_m}} f_{1x_1} \cdots f_{mx_m} \prod_{i=1}^{m} \prod_{j=1}^{n_i} x_i^{(j)}(t_{n_1+\cdots+n_{i-1}+j}) \\ &= \sum_{x\in E_{k,n_1+\cdots+n_m}} \left(\prod_{i=1}^{m} f_{x^{(n_1+\cdots+n_{i-1}+1)}(t_1)\cdots x^{(n_1+\cdots+n_i)}(t_{n_i})} \right) x \end{aligned}$$

as an analytic function over $S \coloneqq S_1 \times \cdots \times S_m \subset \mathbb{R}^{n_1+\cdots+n_m}$, and it satisfies

$$\|f_1 \otimes \cdots \otimes f_m\|_S = \|f_1\|_{S_1} \cdots \|f_m\|_{S_m}.$$

Hence the tensor product determines the bounded multiplication

$$k_{S_1} \hat{\otimes}_k \cdots \hat{\otimes}_k k_{S_m} \to k_S$$

and gives the non-commutative ring structure of

$$\prod_{n=0}^{\infty} k_{[0,1]^n}.$$

Of course we do not have to assume the thickness of the polytopes to define the tensor products, and we just use the assumption for convenience. Recall that the Gel'fand-Neumark theorem (see [DOU]) implies that there is a canonical isometric isomorphism

$$\mathrm{C}^0(X, \mathbb{C}) \otimes_{\max} \mathrm{C}^0(Y, \mathbb{C}) \to \mathrm{C}^0(X \times Y, \mathbb{C})$$

for any compact Hausdorff spaces X and Y, where $\otimes_{\max}$ is the completion of the algebraic tensor product with respect to the maximal tensor norm, which in fact coincides with the minimal tensor norm $\otimes_{\min}$ because a commutative C^*-algebra is always nuclear. We have the direct analogue of this basic fact.

Proposition 1.1.30. *For (thick) polytopes $S_1, \ldots, S_m$ and $S \coloneqq S_1 \times \cdots \times S_m$, the canonical bounded k-algebra homomorphism $k_{S_1} \hat{\otimes}_k \cdots \hat{\otimes}_k k_{S_m} \to k_S$ is an isometric isomorphism. In particular one has the isometric isomorphism*

$$\prod_{n=0}^{\infty} k_{[0,1]^n} \cong_k \prod_{n=0}^{\infty} k_{[0,1]}^{\hat{\otimes} n}$$

of non-commutative k-algebras.

Proof. It is trivial because the restriction

$$k[E_{k,n_1}] \otimes_k \cdots \otimes_k k[E_{k,n_m}] \to k[E_{k,n}]$$

on the dense subalgebras is an isometric isomorphism by the definition of the norms of both sides. □

Now we have a natural Galios action on k_S. This will induce the Galois action on our singular homology of a k-analytic space.

Definition 1.1.31. *For a field K, denote by G_K the absolute Galois group of K.*

Definition 1.1.32 (Galois representation)**.** *For each $g \in G_k$, $x \in \mathbb{Q}^\vee$, and $t \in \mathbb{Q}$, set $(g \cdot x)(t) := g(x(t)) \in k^\times$. The map $g \cdot x \colon \mathbb{Q} \to k^\times \colon t \mapsto (g \cdot x)(t)$ is a group homomorphism, and this determines a $\mathbb{Q}$-linear representation $\mathbb{Q}^\vee \times G_k \to \mathbb{Q}^\vee \colon (x, g) \mapsto g \cdot x$. Take an arbitrary Galois extension K/k of complete non-Archimedean fields contained in $\overline{k}$. Since $K \subset \overline{k}$ is G_k-stable, so is the $\mathbb{Z}$-submodule $\mathbb{Q}_K^\vee \subset \mathbb{Q}^\vee$. Therefore one obtains a G_k-action $\mathbb{Q}_K^\vee \times G_k \to \mathbb{Q}_K^\vee$. This action induces a Galios representation*

$$\begin{array}{rcl} K[E_{K,n}] \times G_k & \to & K[E_{K,n}] \\ \left(\displaystyle\sum_{x \in E_{K,n}} f_x x, g\right) & \mapsto & g\left(\displaystyle\sum_{x \in E_{K,n}} f_x x\right) := \displaystyle\sum_{x \in E_{K,n}} g(f_x)(g \cdot x) = \displaystyle\sum_{x \in E_{K,n}} g(f_{g^{-1} \cdot x}) x \end{array}$$

of a k-algebra for any $n \in \mathbb{N}$. Since $|(g \cdot x)(t)| = |g(x(t))| = |x(t)|$ for any $g \in G_k$, $x \in \mathbb{Q}^\vee$, and $t \in \mathbb{Q}$ by the completeness of k, the action $\mathbb{Q}_K^\vee \times G_k \to \mathbb{Q}_K^\vee$ preserves the map $\| \cdot \|'_S \colon E_{k,n} \to [0, \infty) \colon x \mapsto \|x\|'_S$ for any polytope $S \subset \mathbb{R}^n$. The ideal $I(S) \subset K[E_{k,n}]$ is obviously G_k-stable, and hence the action $K[E_{K,n}] \times G_k \to K[E_{k,n}]$ is isometric with respect to the Gauss seminorm $\| \cdot \|_S$. Thus it induces the isometric Galois representation $K_S \times G_k \to K_S$ of a k-algebra. In particular cosidering the case $K = k$, one has obtained the isometric Galois representaion $k_S \times G_k \to k_S$.

Definition 1.1.33. *The isometric Galois representations $K_S \times G_k \to K_S$ for Galois extensions K/k of complete non-Archimedean fields contained in $\overline{k}$ is extended to an isometric Galois representation $C_S \times G_k \to C_S$ in a natural way. Be careful that $\mathbb{Q}^\vee$ does not coincides with the C-valued character group $Hom_{(group)}(\mathbb{Q}, C^\times)$ unless $k = C$, but the Galois group canonically acts to $Hom_{(group)}(\mathbb{Q}, C^\times)$.*

Lemma 1.1.34. *Let $a \colon S \to T$ be an integral affine map between polytopes $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$. Then the associated bounded C-algebra homomorphism $C_T \to C_S$ is G_k-equivariant.*

Proof. By the continuity of the Galois action, it suffices to show that the k-algebra homomorphism $a^* \colon C[E_{C,n}] \to C[E_{C,m}]$ is G_k-equivariant. Take a presentation $(A, b) \in \mathrm{M}_{m,n}(\mathbb{Z}) \times \mathbb{Z}^m$ of a. For an arbitrary $x \in E_{K,m}$, one has

$$\begin{aligned} g(a^*(x)) &= g\left(\prod_{i=1}^m x^{(i)}(b_i) \prod_{j=1}^n \left(\prod_{i=1}^m \cdot x^{(i)A_{i,j}}(t_j)\right)\right) \\ &= \prod_{i=1}^m g\left(x^{(i)}(b_i)\right) \prod_{j=1}^n \left(\prod_{i=1}^m \left(g \cdot x^{(i)A_{i,j}}\right)(t_j)\right) \\ &= \prod_{i=1}^m \left(g \cdot x^{(i)}\right)(b_i) \prod_{j=1}^n \left(\prod_{i=1}^m \left(g \cdot x^{(i)}\right)^{A_{i,j}}(t_j)\right) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{m} (g \cdot x)^{(i)}(b_i) \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (g \cdot x)^{(i)A_{i,j}}(t_j) \right) \\
&= a^*(g \cdot x)
\end{aligned}
$$

and it follows that $a^* \colon C[E_{C,n}] \to C[E_{C,m}]$ is G_k-equivariant. □

Proposition 1.1.35. *Suppose k is a local field, i.e. a complete discrete valuation field with finite residue field, and let K/k be a finite extension contained in $\overline{k}$. For a polytope $S \subset \mathbb{R}^n$, the G_K-invariants $k_S^{G_K}$ of the G_K-representation k_S coincides with $k \subset k_S$, where G_K is identified as the closed subgroup of G_k corresponding to K. In particular $k_S^{G_k} = k$.*

Proof. The inclusion $k \subset k_S^{G_K}$ is clear.

Since any bounded k-algebra homomorphism associated with an integral affine map is G_k-equivariant, we may and do assume S is thick replacing S to an appropriate polytope. Therefore k_S admits the topological k-basis $E_{k,n} \subset k_S$ by the embedding $k_S \hookrightarrow k^{E_{k,n}}$.

Take an arbitrary function $f \in k_S^{G_K}$. By the assumption $I(S)$, one has the presentation

$$
f = \sum_{x \in E_{k,n}} f_x x.
$$

Fix a real number $r \in (0, \infty)$. By the definition of the Gauss norm $\| \cdot \|_S$, the set $F_r \colon \{x \in E_{k,n} \mid |f_x| \|x\|_S = r\}$ is finite. Since the action of G_k is isometric, the set F_r is stable under the action of G_k. Since F_r is stable under the action of G_K, the stabiliser subgroup $H \subset G_K$ of F_r is a normal subgroup of finite index. Let L/K be the finite extension $\overline{k}^H$. By the definition of the Gauss norm $\| \cdot \|_S$, one has $F_r \neq$ when $r = \|f\|_S$. Suppose F_r is non-empty, and take an element $x \in F_r$. We prove that $x = 1$ and hence $r = |f_1|$. Assume $x \neq 1$. There exists some $i \in \mathbb{N}$ such that $1 \leq i \leq n$ and $x^{(i)} \neq 1 \in \mathbb{Q}^\vee$.

To begin with, we verify $x^{(i)}(t) \in L$ for any $t \in \mathbb{Q}$. Assume $x^{(i)}(t) \notin L$ for some $t \in \mathbb{Q}$. It is clear that $t \neq 0$. Since $x^{(i)}(t) \notin L$, there exists some $g \in H$ such that $(g \cdot x^{(i)})(t) = g(x^{(i)}(t)) neq x^{(i)}(t)$. Then $g(x^{(i)}(t)/x^{(i)}(t)$ is a non-trivial root of unity. Let $a > 1 \in \mathbb{N}$ be the order of $g(x^{(i)}(t))/x^{(i)}(t)$ and $c \in \mathbb{N}_+$ the cardinarity $0 < \#F_r < \infty$ of the non-empty finite set F_r. Take a prime number $l \in \mathbb{N}$ and a sufficiently large integer $N \in \mathbb{N}$ such that $l \mid a$ and $b \coloneqq [L(1^{1/l^N}) : L] > c^2$, where $L(1^{1/l^N})/L$ is the algebraic extension of L generated by primitive l^N-th root of unity. Then $\xi \coloneqq g(x^{(i)}(at/l^N))/x^{(i)}(at/l^N)$ is a primitive l^N-th root of unity. Since $b > c^2 \leq 1$, the primitive l^N-th root ξ of unity is not contained in L and there exist some $g'_1, \ldots, g'_b \in H$ such that $g'_1\xi, \ldots, g'_b\xi$ are distinct primitive l^N-th root of unity. Now one has

$$
\frac{g'_j g(x^{(i)}(at/l^N))}{g'_j x^{(i)}(at/l^N)} = g'_j \xi,
$$

for each $j = 1, \ldots, b$ and hence the set $F' \coloneqq \{xy^{-1} \in E_{k,n} \mid x, y \in F_r\}$ contains b distinct elements. Since $b > c^2$, it contradicts the fact $\#(F_r \times F_r) = c^2$. Consequently $x^{(i)}(t) \in L$ for any $t \in \mathbb{Q}$.

If $|x^{(i)}(1)| \neq 1$, then $|x_i|^{\mathbb{Q}} \subset |L^{\times}| \subset \mathbb{R}_+^{\times}$, and it contradicts the fact that L is a discrete valuation field. Therefore one has $|x_i(1)| = 1$. It follows that $x_i(q_L - 1) = x_i(1)^{q_L - 1} \in 1 + m_L \subset O_L^{\times}$, where O_L is the ring of integers in L, $m_L \subset O_L$ is the maximal ideal of O_L, and $q_L \in \mathbb{N}$ is the cardinarity $\# \tilde{L}$ of the finite residue field $\tilde{L}$. We show that $x^{(i)}(q_L - 1) = 1$. Assume $x^{(i)}(q_L - 1) \neq 1$. Take the greatest integer $d \in \mathbb{N}_+$ such that $x^{(i)}(q_L - 1) - 1 \in m_L^d$. Since $x^{(i)}(q_L - 1) \neq 1$, one has $x^{(i)}((q_L - 1)p^{-d-1}) \in 1 + m_L \backslash \{1\}$ and hence

$$x^{(i)}(q_L - 1) = x^{(i)}((q_L - 1)p^{-d-1})^{p^{d+1}} \in 1 + m_L^{d+1}.$$

It contradicts the maximality of d, and we conclude $x^{(i)}(q_L - 1) = 1$.

Let $e \in \mathbb{N}_+$ be the order of the torsion element $x^{(i)}(1) \in L^{\times}$. Since $x^{(i)} \neq 1$, we know $e > 1$ and hence $e \mid q_L - 1$ has a prime factor $l' \neq p \in \mathbb{N}$. The image of l' in the Abelian group $\mathbb{Z}/e\mathbb{Z}$ is a torsion element and therefore $x^{(i)}(l'^{-m}) \in L$ is a primitive (el'^m)-th root of unity for any $m \in \mathbb{N}$. It follows that the finite field $\tilde{L}$ has infinitely many root of unity, and it is contradiction. We conclude $x = 1$. Since the choice of r and x was arbitrary, one has obtained that $f = f_1 \in k$. □

Corollary 1.1.36. *In the same situation above, if K/k is a Galois extension, the G_k-invariants $K_S^{G_k}$ of the G_k-representation K_S coincides with the k.*

Proof. Since G_k contains G_K, the invariants $K_S^{G_k}$ is contained in $K_S^{G_K} = K$. The statement follows because $K^{G_k} = k$. □

Corollary 1.1.37. *In the same situation above, the G_k-invariants $(k_S \otimes_k \overline{k})^{G_k}$ of the G_k-representation $k_S \otimes_k \overline{k}$ coincides with k, where the action of G_k on $k_S \otimes_k \overline{k}$ is given by setting $g(f \otimes a) := g(f) \otimes g(a)$ for each $g \in G_k$, $f \in k_S$, and $a \in \overline{k}$.*

Proof. One has

$$k_S \otimes_k \overline{k} = k_S \otimes_k \varinjlim_{\overline{k}/K/k} K = \varinjlim_{\overline{k}/K/k} k_S \otimes_k K = \varinjlim_{\overline{k}/K/k} k_S \hat{\otimes}_k K \subset \varinjlim_{\overline{k}/K/k} K_S,$$

where K in the limit runs through all finite subextension $\overline{k}/K/k$. These identifications are compatible with the action of G_k, and hence we have done. □

Now we see the properties of the ring k_S of analytic functions. The norm on k_S is not multiplicative, and hence k_S can not be embedded in a valuation field. However, it is easily proved that k_S is an integral domain and its norm is powermultiplicative.

Definition 1.1.38. *A seminormed k-algebra is said to be uniform ([BER1]) or uniformly seminormed if the seminorm is power-multiplicative.*

Proposition 1.1.39. *For a polytope $S \subset \mathbb{R}^n$, the k-Banach algebra k_S is a uniformly normed integral domain.*

Proof. We may and do assume S is thick and hence $S \cap \mathbb{Q}^n \neq \emptyset$. Embedding k_S in C_S, it suffices to prove the integrality in the case k is algebraically closed. The parallel translation $k_S \to k_{S-a} \colon f(t) = \sum f_x x \mapsto f(t+a) = \sum f_x x(a) x$ is isometric isomorphism for any $a \in \mathbb{Q}^n$, because $-a \colon S \to S - a \colon s \mapsto s - a$ is an isomorphic affine map. Therefore we may and do assume S contains $(0, \ldots, 0) \in \mathbb{R}^n$. Then one has $\|x\|_S \geq 1$ for any $x \in E_{k,n}$ because $x(0, \ldots, 0) = 1$. We use the assumption $S \cap \mathbb{Q}^n \neq \emptyset$ only for this inequality $\|x\|_S \geq 1$.

Take an arbitarary pair $f, g \in k_S \backslash \{0\}$, and we prove $fg \neq 0$. Since $\lim_{x \in E_{k,n}} |f_x| \|x\|_S = 0$ and $|f_x| \|x\|_S \geq |f_x|$ for any $x \in E_{k,n}$, one has $\lim_{x \in E_{k,n}} |f_x| = 0$. Similarly we know $\lim_{x \in E_{k,n}} |g_x| = 0$. Hence the sets $\{|f_x| \mid x \in E_{k,n}\}$ and $\{|g_x| \mid x \in E_{k,n}\}$ are bounded in $\mathbb{R}$, and $\max\{|f_x| \mid x \in E_{k,n}\}$ and $\max\{|g_x| \mid x \in E_{k,n}\}$ exist. Set $F \coloneqq \{x \in E_{k,n} \mid |f_x| = \max\{|f_{x'}| \mid x' \in E_{k,n}\}$ or $|g_x| = \max\{|g_{x'}| \mid x' \in E_{k,n}\}\}$, and then F is a non-empty finite set. Let $M \subset E_{k,n}$ be the finitely generated non-trivial $\mathbb{Z}$-submodule generated by F, and fix a total order of M compatible with its group-structure, i.e. give it the structure of a totally ordered group. The existence of a total order follows from the existence of $\mathbb{Z}$-basis of M by the structure theorem of a finitely generated Abelian group. Give it the lexicographic order with respect to the presentation by the basis. Take the least elements x_1 and x_2 from the set $\{x \in M \mid |f_x| = \max\{|f_{x'}| \mid x' \in E_{k,n}\}\}$ and $\{x \in M \mid |g_x| = \max\{|g_{x'}| \mid x' \in E_{k,n}\}\}$ which are not empty by the definition of M, with respect to the fixed total order. Then $|(fg)_{x_1 x_2}| = |f_{x_1}||g_{x_2}|$. Indeed, take arbitrary $x, y \in E_{k,n}$ satisfying $xy = x_1 x_2$. If $x \notin M$ or $y \notin M$, then $|f_x||g_y| < |f_{x_1}||g_{x_2}|$ by the denifinition of M. On the other hand, suppose $x, y \in M$. If x is smaller than x_1 with respect to the total order of M, then $|f_x| < |f_{x_1}|$ by the definition of x_1, and hence $|f_x||g_y| < |f_{x_1}||g_{x_2}|$. Similarly if y is smaller than x_2, then one has $|f_x||g_y| < |f_{x_1}||g_{x_2}|$. If x is larger than x_1, then y is smaller than x_2 because the total order of M is compatible with the group-structure of M and $xy = x_1 x_2$. Therefore $|f_x||g_y| < |f_{x_1}||g_{x_2}|$. Similarly if y is larger than x_2, then x is smaller than x_1 and $|f_x||g_y| < |f_{x_1}||g_{x_2}|$. In conclusion, one has $|f_x||g_y| < |f_{x_1}||g_{x_2}|$ for any $(x, y) \neq (x_1, x_2)$ satisfying $xy = x_1 x_2$, and it follows $|(fg)_{x_1 x_2}| = |f_{x_1}||g_{x_2}| > 0$. Consequently $fg \neq 0$. The proof of the power-multiplicativity of the norm $\| \cdot \|_S$ is repetition of a similar argument considering the finite subset $\{x \in E_{k,n} \mid |f_x| \|x\|_S = \|f\|_S\}$ for each $f \in k_S$ instead of the subset $\{x \in E_{k,n} \mid |f_x| = \max\{|f_{x'}| \mid x' \in E_{k,n}\}\}$.

Take an arbitrary analytic function $f \in k_S$, and it suffices to show that $\|f^2\| = \|f\|^2$. Since $\lim_{x \in E_{k,n}} |f_x| \|x\|_S = 0$, the set $\{|f_x| \|x\|_S \mid x \in E_{k,n}\}$ is bounded in $\mathbb{R}$, and $\max\{|f_x| \|x\|_S \mid x \in E_{k,n}\}$ exists. Set $F \coloneqq \{x \in E_{k,n} \mid |f_x| \|x\|_S = \|f\|_S\}$, and then F is a finite set. Let $M \subset E_{k,n}$ be the finitely generated $\mathbb{Z}$-submodule generated by F, and fix a total order of M compatible with its group-structure. Take the least element x_1 of F with respect to the fixed total order of M. Then $|(f^2)_{x_1^2}| \|x_1^2\|_S = |f_{x_1}|^2 \|x_1\|_S^2$. Indeed, take arbitrary $x, y \in E_{k,n}$ satisfying $xy = x_1^2$. If $x \notin M$ or $y \notin M$, then $|f_x f_y| \|xy\|_S \leq |f_x||f_y| \|x\|_S \|y\|_S < |f_{x_1}|^2 \|x_1\|_S^2$ by the denifinition of M. On the other hand, suppose $x, y \in M$. If x is smaller than x_1 with respect to the total order of M, then $|f_x| \|x\|_S < |f_{x_1}| \|x_1\|_S$ by the definition of x_1, and hence $|f_x f_y| \|xy\|_S \leq |f_x||f_y| \|x\|_S \|y\|_S < |f_{x_1}|^2 \|x_1\|_S^2$. Similarly if y is smaller than x_1, then one has $|f_x f_y| \|xy\|_S \leq |f_x||f_y| \|x\|_S \|y\|_S < |f_{x_1}|^2 \|x_1\|_S^2$. If x is larger than x_1, then y is smaller than x_2 because the total order of M is compatible with the group-structure of M

and $xy = x_1^2$. Therefore $|f_x f_y| \|xy\|_S < |f_{x_1}|^2 \|x_1\|_S^2$. Similarly if y is larger than x_2, then x is smaller than x_1 and $|f_x f_y| \|xy\|_S < |f_{x_1}|^2 \|x_1\|_S^2$. In conclusion, one has $|f_x f_y| \|xy\|_S < |f_{x_1}|^2 \|x_1\|_S^2 = |(f_{x_1})^2| \|x_1^2\|_S$ for any $(x, y) \neq (x_1, x_1)$ satisfying $xy = x_1^2$, and it follows $|(f^2)_{x_1^2}| \|x_1^2\|_S = |(f_{x_1})^2| \|x_1^2\|_S$. Consequently $\|f^2\|_S = \|f\|_S^2$. $\square$

At the end of the introduction of the basic properties of the k-Banach algebra k_S, we give an involution. This involution induces an involution of the analytic singular homology in §3.1, and it plays an important role as a reflection of the coordinate.

Definition 1.1.40 (involution)**.** *For an integer $m \in \mathbb{N}$, define the involution $*\colon k_{[0,m]} \to k_{[0,m]}$ by*

$$\begin{array}{rcl} *\colon k_{[0,m]} & \to & k_{[0,m]} \\ f = \displaystyle\sum_{x \in E_{k,1}} f_x x & \mapsto & f^* := \displaystyle\sum_{x \in E_{k,1}} f_{x^{-1}} x(-m) x, \end{array}$$

which is the isometric isomorphism associated with the isomorphic integral affine map $*\colon [0, m] \to [0, m]\colon t \mapsto m - t$. Since $2 \in k^\times$, one has the canonical decomposition $k_{[0,m]} = \mathfrak{R} k_{[0,m]} \oplus \mathfrak{I} k_{[0,m]}$ into the eigen spaces, where $\mathfrak{R} k_{[0,m]}$ is the "real analytic functions" $\{f \in k_{[0,m]} \mid f^* = f\}$ containing k and $\mathfrak{I} k_{[0,m]}$ is the "imaginary analytic functions" $\{f \in k_{[0,m]} \mid f^* = -f\}$. Note that the invariants $\mathfrak{R} k_{[0,m]} \subset k_{[0,m]}$ contains k and is a closed k-subalgebra. The involution $*\colon k_{[0,m]} \to k_{[0,m]}$ induces a topological involution $*\colon [0, m]_k \to [0, m]_k$ whose restriction on $[0, 1] \subset [0, 1]_k$ is the reflection $*\colon [0, m] \to [0, m]\colon t \mapsto m - t$. For each $x \in E_{k,1}$, set $\overline{x} = x^* \in k[E_{k,1}]$ and call it the conjugation of x.

1.2 Non-Archimedean analytic realisation of a polytope

We introduce the notion of the "non-Archimedean realisation" of a polytope which lives in the Euclidean world. The non-Archimedean realisations of polytopes such as the unit interval $[0, 1]$ will be used in the construction of the analytic singular homology in §3.1.

Definition 1.2.1. *Let $S \in \mathbb{R}^n$ be a polytope. Denote by S_k the non-empty compact Hausdorff space $\mathscr{M}(k_S)$, where $\mathscr{M}(k_S)$ is Berkovich's spectrum of k_S as a k-Banach algebra in the sense of [BER1].*

The non-emptiness, compactness, and Hausdorffness of $\mathscr{M}(k_S)$ follow from the general fact of a commutative k-Banach algebra (see [BER1]). We formally set $\emptyset_k := \emptyset$. We use the spectrum S_k instead of a subset S in our theory of homology and integral, and recognise S_k as the realisation of S in the non-Archimedean world.

Definition 1.2.2 (ground field extension)**.** *For a polytope $S \subset \mathbb{R}^n$ and an extension K/k of complete non-Archimedean fields, the ground field extension map $k_S \hookrightarrow K_S$ induces the continuous map $S_K \to S_k$ which is compatible with the embeddings $S \subset S_k, S_K$. Also call it the ground field extension map. This ground field extension map induces a continuous map $\mathbb{R}^n_K \to \mathbb{R}^n_k$. Also call it the ground field extension map.*

Lemma 1.2.3. *Let $S \in \mathbb{R}^n$ be a polytope such that $S \cap \mathbb{Q}^n \neq \emptyset$. Then the spectrum S_k is connected.*

Proof. It directly follows from Proposition 1.1.39 and Shilov Idempotent theorem in [BER1], 7.4.1. □

Since we do not know whether an arbitrary rational domain $r \subset S_k$ in the natural sense has infinitely many connected components or not, the local connectedness and the arcwise connectedness of the spectrum S_k is not verified. Note that a connected, locally connected, metrisable compact space is arcwise connected and locally arcwise connected by [ENG], 6.3.11. and 6.3.12. and therefore we know any connected analytic space is arcwise connected. Now the realisation $S \rightsquigarrow S_k$ preserves the inclusion as below:

Lemma 1.2.4. *Let $S \subset T$ be polytopes. Then the continuos map $S_k \to T_k$ induced by the restriction map $k_T \to k_S$ associated with the inclusion $S \hookrightarrow T$ is a homeomorphism onto its image.*

Proof. In order to show the injectivity, it suffices to show that the image of k_S in k_T is dense, and it follows from the fact that the image of $k[E_{k,n}]$ is dense in both of k_S and k_T. In addition, a continuous injective map from a compact space to a Hausdorff space is a homeomorphism onto its image. □

Definition 1.2.5. *Denote by Top the categpry of topological spaces whose morphisms are continuous maps. Denote by (k-Banach) the category of k-Banach algebras whose morphisms are bounded k-algebra homomorphisms. Denote by (Polytope) the category of polytopes whose morphisms are integral affine maps.*

We identify S_k as a subspace of T_k through this homeomorphism onto the image. The realisation $S \rightsquigarrow S_k$ is obtained by the composition of the correspondences given by the functors

$$\begin{array}{rcl} (\text{Polytope}) & \to & (k\text{-Banach}) \\ S & \rightsquigarrow & k_S \\ a & \rightsquigarrow & a^* \end{array}$$

and

$$\begin{array}{rcl} (k\text{-Banach}) & \to & \text{Top} \\ A & \rightsquigarrow & \mathscr{M}(A) \\ \phi & \rightsquigarrow & \phi^*, \end{array}$$

and hence regard it as the realisation functor

$$\begin{array}{rcl} (\text{Polytope}) & \to & \text{Top} \\ S & \rightsquigarrow & S_k \end{array}$$

$$a \rightsquigarrow a^{**}.$$

In particular an isomorphic integral affine map $a\colon S \to T$ induces a homeomorphism $a^{**}\colon S_k \to T_k$.

Though we have defined S_k only for a polytope $S \in \mathbb{R}^n$, one obtains $\mathbb{R}^n_k$ by gluing spectra corresponding to thick polytopes. The space $\mathbb{R}^n_k$ contains the affine space $\mathbb{R}^n$.

Definition 1.2.6. *For an integer $n \in \mathbb{N}$, denote by $P_n \subset 2^{\mathbb{R}^n}$ the set of thick polytopes $S \subset \mathbb{R}^n$. Note that for any $S, T \in P_n$, setting $U \subset \mathbb{R}^n$ as the convex closure of the union of S and T, U satisfies $U \in P_n$ and $S, T \subset U$. Therefore P_n is a directed endowed with the semiorder given by the inclusion relation $\subset$. Set*

$$\mathbb{R}^n_k := \varinjlim_{S \in P_n} S_k,$$

where the limit is the direct limit in the category Top. Note that since P_n is final in the set of polytopes embedded in $\mathbb{R}^n$, the direct limit does not differ if one replaces P_n to the set of polytopes embedded in $\mathbb{R}^n$.

The direct limit space $\mathbb{R}^n_k$ is a Lindelöf Hausdorff space, and hence paracompact and normal. The Lindelöfness follows from the fact that $\mathbb{R}^n_k$ can be presented as a countable direct limit $\varinjlim_{m \in \mathbb{N}}([-m, m]^n)_k$ of compact Hausdorff spaces. The Hausdorffness is verified in the following way: Take arbitrary distinct points $t_1, t_2 \in \mathbb{R}^n_k$. Since the underlying set of $\mathbb{R}^n_k$ is the set-theoretical injective limit, there exists some non-empty bounded subset $S \subset \mathbb{R}^n$ such that $t_1, t_2 \in S_k$. The density of $k[E_{k,n}]$ in k_S gurantees that the restrictions $t_1|_{k[E_{k,n}]}$ and $t_2|_{k[E_{k,n}]}$ differs from each other, and there exists some $f \in k[E_{k,n}]$ such that $t_1(f) \neq t_2(f)$. We may assume $t_1(f) < t_2(f)$ without loss of generality. Then the subsets

$$\begin{aligned} U_1 &:= \{t \in \mathbb{R}^n_k \mid t(f) < t_1(f)/2 + t_2(f)/2\} \\ \text{and } U_2 &:= \{t \in \mathbb{R}^n_k \mid t(f) > t_1(f)/2 + t_2(f)/2\} \end{aligned}$$

are disjoint subsets containing t_1 and t_2 respectively, and are open subsets by the definition of the topology of S_k's and the direct limit topology of $\mathbb{R}^n_k$.

Remark that $\mathbb{R}^n_k$ is properly contained in the set of all multiplicative seminorm on $k[E_{k,n}]$ whose restriction on k coincides with the norm of k. Now there are two canonical ways to embed S and $S \cap \mathbb{Q}^n$ into S_k:

Definition 1.2.7. *Let $S \in \mathbb{R}^n$ be a thick polytope. Define the set-theoretical maps $i_u\colon S \cap \mathbb{Q}^n \to S_k$ and $i_p\colon S \to S_k$ in the following way:*

$$\begin{aligned} i_u\colon S \cap \mathbb{Q}^n &\to S_k \\ t &\mapsto i_u(t),\ \ |f(i_u(t))| := \left| \sum_{x \in E_{k,n}} f_x x(t) \right| \\ i_p\colon S &\to S_k \end{aligned}$$

$$t \quad \mapsto \quad i_p(t), \quad |f(i_p(t))| \coloneqq \max\left\{ |f_x||x(t)| \;\middle|\; x \in E_{k,n} \right\},$$

where the infinite sum $\sum_{x \in E_{k,n}} f_x x(t)$ in the definition of i_u is the convergent limit as an element of the completion C of $\overline{k}$, because for any $f \in k_S$ one has

$$0 \leq \lim_{x \in E_{k,n}} |f_x||x(t)| \leq \lim_{x \in E_{k,n}} |f_x| \|x\|_S = 0.$$

Definition 1.2.8. *Let $S \in \mathbb{R}^n$ be a polytope. Take a polytope $T \subset \mathbb{R}^m$ such that $I(T) = 0$ and k_S is isometrically isomorphic to k_T through the bounded k-algebra homomorphism associated with an isomorphic integral affine map $a \colon S \to T$. Define the set-theoretical maps $i_u \colon S \cap \mathbb{Q}^n \to S_k$ and $i_p \colon S \to S_k$ in the following way:*

$$\begin{aligned} i_u \colon S \cap \mathbb{Q}^n &\xrightarrow{a} T \cap \mathbb{Q}^n \xrightarrow{i_u} T_k \xrightarrow{(a^{**})^{-1}} S_k \\ i_p \colon S &\xrightarrow{a} T \xrightarrow{i_p} T_k \xrightarrow{(a^{**})^{-1}} S_k. \end{aligned}$$

Then obviously they are independent of the choice of the polytope T and the isomorphic integral affime map a, and these maps commute with the inclusions $S \subset S' \subset \mathbb{R}^n$ and $S_k \subset S'_k \subset \mathbb{R}^n_k$. Indeed, for a point $t \in \mathbb{R}^n$ the image $i_p(t) \in S_k$ corresponds to the multiplicative norm which is the unique continuos extension of the quotient norm on $k[E_{k,n}]/I(S) \subset k_S$ of the multiplicative norm

$$\begin{aligned} k[E_{k,n}] &\to [0, \infty) \\ f &\mapsto \max_{x \in E_{k,n}} |f_x||x(t)|, \end{aligned}$$

and for a point $t \in \mathbb{Q}^n$ the image $i_u(t) \in S_k$ corresponds to the multiplicative seminorm which is the unique continuos extension of the quotient norm on $k[E_{k,n}]/I(S) \subset k_S$ of the multiplicative norm

$$\begin{aligned} k[E_{k,n}] &\to [0, \infty) \\ f &\mapsto \left| \sum_{x \in E_{k,n}} f_x x(t) \right|. \end{aligned}$$

Proposition 1.2.9. *The ground field extension map $S_K \to S_k$ is compatible with the embeddings $i_p \colon S \hookrightarrow S_k, S_K$.*

Proof. Trivial by the definition of i_p. □

Proposition 1.2.10. *For a polytope $S \subset \mathbb{R}^n$, the map i_p is a homeomorphism onto its image, but the image of i_u is a discrete subspace contained in C-rational points $S_k(C)$. The image $i_p(\mathbb{Z}^n) \subset S_k(C)$ is contained in k-rational points $S_k(k)$. Moreover, the image of i_p is closed.*

Proof. It suffices to show it in the case S is thick.

To begin with, we have to check $i_u(t)$ and $i_p(t)$ are points of the spectrum S_k for any $t \in S$, i.e. they are multiplicative bounded seminorms on k_S. The multiplicativity of $i_u(t)$ follows from that of the norm $|\cdot|$ of C. We show the multiplicativity of $i_p(t)$. Take an arbitrary $f, g \in k_S$, and let F be the finite set $\{x \in E_{k,n} \mid |f_x||x(t)| = |f(i_p(t))| \text{ or } |g_x||x(t)| = |g(i_p(t))|\}$, which is non-empty because $0 \leq \lim_{x \in E_{k,n}} |f_x||x(t)| \leq \lim_{x \in E_{k,n}} |f_x|\|x\|_S = 0$. By the similar argument in Proposition 1.1.39 with a total order of the $\mathbb{Z}$-submodule of $E_{k,n}$ generated by F, one obtains $|f(i_p(t))||g(i_p(t))| = |fg(i_p(t))|$ because $|x(t)||y(t)| = |xy(t)|$ for any $x, y \in E_{k,n}$. Hence $i_p(t)$ is multiplicative.

Now we begin the proof of the proposition. Each point in the image of i_p is obviously a C-rational point, and i_p sends a point on $\mathbb{Z}^n$ to a k-rational point by the definition of i_p.

First, we prove that $i_u(S \cap \mathbb{Q}^n)$ is discrete. Take an arbitrary $t = (t^{(1)}, \ldots, t^{(n)}) \in S \cap \mathbb{Q}^n$ and set $t^{(i)} = q_i/r_i$, $q_i, r_i \in \mathbb{Z}$, and $r_i \neq 0$. Denote by $\underline{p} \in \mathbb{Q}_k^\vee$ the system of power roots of p, i.e. $\underline{p}(1) = p$, set $\underline{p}_i := (\underline{p}_i^{(1)} = 1, \ldots, \underline{p}_i^{(i-1)} = 1, \underline{p}_i^{(i)} = \underline{p}, \underline{p}^{(i+1)} = 1, \ldots, \underline{p}_i^{(n)} = 1) \in E_{k,n} \subset k_S$ for each $i = 1, \ldots, n$. By the definition of the topology of the spectrum, the subset $U \equiv \{t' \in S_k \mid |(p^{-q_i}\underline{p}_i^{r_i} - 1)(t')| < 1, i = 1, \ldots, n\} \cap \{t' \in S_k \mid |(p^{q_i}\underline{p}_i^{-r_i} - 1)(t')| < 1, i = 1, \ldots, n\}$ is an open neighbourhood of t and contains $i_u(t')$ for no $t' \neq t \in S \cap \mathbb{Q}^n$. Hence $i_u(S \cap \mathbb{Q}^n)$ is discrete.

Secondly, we prove the continuity of i_p for an arbitrary bounded subset S. By the definition of the topology of the spectrum, it suffices to show that the pre-image of an open subset U of the form $\{t \in S_k \mid |f(t)| < M\}$ or $\{t \in S_k \mid L < |f(t)| < M\}$ for an analytic function $f \in k_S$ and real numbers $L < M \in (0, \infty)$ by i_p is open in S. Set $U := \{t \in S_k \mid |f(t)| < M\}$. Since $k[E_{k,n}]$ is dense in k_S, there exists some $f' \in k[E_{k,n}]$ such that $|f - f'| < M$. Then $U = \{t \in S_k \mid |f(t)| < M\} = \{t \in S_k \mid |f'(t)| < M\}$, and hence we replace f to f'. Set $f = a_1x_1 + \cdots + a_mx_m$ by some $a_1, \ldots, a_m \in k$ and $x_1, \ldots, x_m \in E_{k,n}$. Take an arbitrary point $t \in S$ contained in the pre-image of U. Since $|a_i||x_i(t)| \leq |f(i_p(t))| < M$, the pre-image of U contains some open neighbourhood of t in S, because of the continuity of the map $t \mapsto |x_i(t)|$. Therefore the pre-image of U is open in S in this case. Set $U \equiv \{t \in S_k \mid L < |f(t)| < M\}$ this time. We replace f to $f' \in k[E_{k,n}]$ again, and hence f can be presented as $a_1x_1 + \cdots + a_mx_m$ by some $a_1, \ldots, a_m \in k$ and $x_1, \ldots, x_m \in E_{k,n}$ also in this time. Take an arbitrary point $t \in S$ contained in the pre image of U. By the definition of i_p, there exists some $1 \leq i \leq m$ such that $|a_i||x_i(t)| = |f(i_p(t))|$, and set $F \equiv \{i \mid |a_i||x_i(t)| = |f(i_p(t))|\} \neq \emptyset$. Since the map $t \mapsto |x_i(t)|$ is continuous for each $1 \leq i \leq m$, there exists some open neighbourhood $V \subset S$ of t such that for any $t' \in V$, $L < |a_i||x_i(t')| < M$ for any $i \in F$ and $|a_i||x_i(t')| < M$ for any $i \notin F$. Then obviously V is contained in the pre-image of U, and therefore the pre-image of U is open in S also in this case. It implies that i_p is continuous. Since a polytope is compact, the image $i_p(S) \subset S_k$ is also compact. Therefore it is closed because S_k is Hausdorff.

Finally, we prove that i_p is an injective open map onto its image. The injectivity is trivial. Indeed, for any $t = (t_1, \ldots, t_n), t' = (t'_1, \ldots, t'_n) \in S$, suppose $t \neq t'$. Take an $1 \leq i \leq n$ such that $t_i \neq t'_i$. Set $\underline{p}_i \equiv (\underline{p}_i^{(1)} = 1, \ldots, \underline{p}_i^{(i-1)} = 1, \underline{p}_i^{(i)} = \underline{p}, \underline{p}^{(i+1)} = 1, \ldots, \underline{p}_i^{(n)} = 1) \in E_{k,n} \subset k_S$ for this i and then one has $|(\underline{p}_i(-t)\underline{p}_i)(i_p(t))| = 1$ and $|(\underline{p}_i(-t)\underline{p}_i)(i_p(t'))| =$

$|\underline{p}_i(t'_1 - t_1, \ldots, t'_n - t_n)| = p^{t'_i - t_i} \neq 1$. This implies $i_p(t) \neq i_p(t')$. We prove the openness of the map i_p. It suffices to show that for any $a_1 < b_1, a_2 < b_2, \ldots, a_n < b_n \in \mathbb{R}$, the image of $((a_1, b_1) \times \cdots \times (a_n, b_n)) \cap S$ in $i_p(S)$ is open. Set $\underline{p}_i \equiv (\underline{p}_i^{(1)} = 1, \ldots, \underline{p}_i^{(i)} = \underline{p}, \ldots, \underline{p}_i^{(n)} = 1) \in E_{k,n}$ for each $1 \leq i \leq n$. Then the open subset $((a_1, b_1) \times \cdots \times (a_n, b_n)) \cap S$ can be presented as $\{t \in S \mid p^{a_i} < |\underline{p}_i(i_p(t))| < p^{b_i}, 1 \leq i \leq n\}$, and therefore the image of it is presented as $\{t \in i_p(S) \mid p^{a_i} < |\underline{p}_i(t))| < p^{b_i}, 1 \leq i \leq n\}$. Obviously it's open in $i_p(S)$ by the definition of the topology of the spectrum. $\square$

Definition 1.2.11. *Define maps $i_u \colon \mathbb{Q}^n \to \mathbb{R}_k^n$ and $i_p \colon \mathbb{R}^n \to \mathbb{R}_k^n$ as the direct limits*

$$i_u \colon \mathbb{Q}^n = \varinjlim_{S \in P_n} S \cap \mathbb{Q}^n \xrightarrow{i_u} \varinjlim_{S \in P_n} S_k = \mathbb{R}^n$$

$$\text{and} \qquad i_p \colon \mathbb{R}^n = \varinjlim_{S \in P_n} S \xrightarrow{i_p} \varinjlim_{S \in P_n} S_k = \mathbb{R}^n.$$

By the proposition above, we regard S as a subspace of S_k by i_p. Particularly, we have brought the standard simplex Δ^n, the cube $[0,1]^n$, and the real affine space $\mathbb{R}^n$ in the non-Archimedean world. The point $i_p(t)$ corresponding to a real vector $t \in \mathbb{R}^n$ determines a complete valuation field. If $t \in \mathbb{Q}^n$, then the singleton $\{t\} \subset \mathbb{R}^n$ is a polytope but the valuation field does not coincide with the competion of the field of fraction of $k_{\{t\}}$, because $I(\{t\}) \neq 0$ and $k_{\{t\}} \subset \overline{k}$. We introduce the k-Banach algebra k_t corresponding to the point $t \in \mathbb{R}^n$.

Definition 1.2.12. *For each $t \in \mathbb{R}^n$, denote by k_t the completion of $k[E_{k,n}]$ by the multiplicative seminorm corresponding to $i_p(t) \in \mathbb{R}_k^n$, and denote by $Q(k_t)$ the field of fraction of k_t. Set $t_k \coloneqq \mathscr{M}(k_t)$, and denote by $\| \cdot \|_t$ the multiplicative norm of k_t.*

Remark that when t is contained in some thick polytope $S \subset \mathbb{R}^n$, then the spectrum t_k coincides with the subspace of S_k consisting of norms smaller than or equal to $i_p(t) \in S_k$ because $\| \cdot \|_t = i_p(t)$. Note that since each element of the image of $E_{k,n}$ is invertible in k_S, one has $t'(x) = i_p(t)(x)$ for any $t' \in t_k$ and $x \in E_{k,n}$. Be careful of the ambiguity of $i_p(t)$. If $S \subset \mathbb{R}^n$ is thick, then the images of $t \in S \subset \mathbb{R}^n$ by the embeddings $i_p \colon S \to S_k \subset \mathbb{R}_k^n$ and $i_p \colon \mathbb{R}^n \to \mathbb{R}_k^n$ trivially coincide by the definition of them, but if $S \subset \mathbb{R}^n$ is not thick, then they are distinct. The image of t by $i_p \colon S \to S_k \subset \mathbb{R}_k^n$ determines a seminorm on $k[E_{k,n}]$ whose kernel is $J(S) \neq 0$, while the image of t by $i_p \colon \mathbb{R}^n \to \mathbb{R}_k^n$ is a norm. When we write "$i_p(t) \in S_k$" for a point $t \in S$, we mean the image of t by $i_p \colon S \to S_k$ but not the image of t by $i_p \colon \mathbb{R}^n \to \mathbb{R}_k^n$. This notation is never ambiguous because the latter image is not contained in S_k when $S \subset \mathbb{R}_k^n$ is not thick.

If $t = (t_1, \ldots, t_n) \in \mathbb{Q}^n$, the spectrum t_k contains not only $i_p(t)$ but also $i_u(t)$ because $i_u(t) \leq i_p(t)$. More generally, we can take infinitely many C-rational points from t_k by the following way, where C is the completion of $\overline{k}$. Suppose $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = 1$ first, where $\sqrt{|k^\times|}$ is the $\mathbb{Q}$-vector subspace of $\mathbb{R}_+^\times$ generated by the image $|k^\times|$ of the norm $| \cdot | \colon k^\times \to \mathbb{R}_+^\times$. Take an arbitrary $\mathbb{Q}$-basis $\{x_\lambda \mid \lambda \in \Lambda\}$ of the $\mathbb{Q}$-vector subspace $V \subset E_{C,n}$ consisting of

elements $x = (x_1, \ldots, x_n)$ satisfying $|x_1(1)| = \cdots = |x_n(1)| = 1$. Take a system $\underline{p} \in \mathbb{Q}_k^\vee$ of power roots of p, and set $\underline{p}^{(i)} \equiv (\underline{p}_1^{(i)} = 1, \ldots, \underline{p}_i^{(i)} = \underline{p}, \ldots, \underline{p}_n^{(i)} = 1) \in E_{k,n}$ for each $i = 1, \ldots, n$. Then one has

$$E_{C,n} = V \oplus \left(\bigoplus_{i=1}^{n} (\underline{p}^{(i)})^{\mathbb{Q}} \right) = \left(\bigoplus_{x_\lambda \in \Lambda} x_\lambda^{\mathbb{Q}} \right) \oplus \left(\bigoplus_{i=1}^{n} (\underline{p}^{(i)})^{\mathbb{Q}} \right),$$

by the assumption $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = 1$. Denote by Λ' the index set $\Lambda \sqcup \{\underline{p}^{(1)}, \ldots, \underline{p}^{(n)}\}$ of the $\mathbb{Q}$-basis $\{x_\lambda \mid \lambda \in \Lambda \sqcup \{\underline{p}^{(1)}, \ldots, \underline{p}^{(n)}\}\}$ of $E_{C,n}$, where $x_{\underline{p}^{(i)}} := \underline{p}^{(i)}$ for each $i = 1, \ldots, n$. For each $u = (u_{x_\lambda})_{x_\lambda \in \Lambda'} \in (C^{\circ\times})^{\Lambda'}$, define the C-valued character $\phi_u \colon C_t \to C$ by

$$\phi_u(f) := \sum_{I \in \mathbb{Q}^{\oplus \Lambda'}} f_{x^I} \underline{p}(I_{\underline{p}^{(1)}} t_1 + \cdots + I_{\underline{p}^{(n)}} t_n) u^I,$$

where x^I and u^I is the essentially finite product $\Pi x_\lambda^{I_\lambda} \in E_{C,n}$ and $\Pi u_\lambda^{I_\lambda} \in C^{\circ\times}$. The infinite sum in the right side hand converges because

$$\begin{aligned} \left| f_{x^I} \underline{p} \left(I_{\underline{p}^{(1)}} t_1 + \cdots + I_{\underline{p}^{(n)}} t_n \right) u^I \right| &= |f_{x^I}| |\underline{p}|^{I_{\underline{p}^{(1)}} t_1 + \cdots + I_{\underline{p}^{(n)}} t_n} = |f_{x^I}| \left| \underline{p}^{I_{\underline{p}^{(1)}}} \right|^{t_1} \cdots \left| \underline{p}^{I_{\underline{p}^{(n)}}} \right|^{t_n} \\ &= |f_{x^I}| \| x^I \|_t \to 0. \end{aligned}$$

This is a bounded character, and therefore determines a multiplicative seminorm on C_t. Restricting it on $k_t \subset C_t$, one obtains a C-rational multiplicative seminorm on k_t. Though the restriction is not injective, this way gives infinitely many C-rational points in t_k. If in general $\dim_{\mathbb{Q}} \sqrt{|k^\times|} \neq 1$, considering a lift in $\overline{k}^\times$ of an arbitrary $\mathbb{Q}$-basis of $\sqrt{|k^\times|}$ instead of p by the norm $|\cdot| \colon \overline{k}^\times \to \mathbb{R}_+^\times$, one obtains various C-rational points of t_k in the same way.

Although i_u does not preserve the topology of $\mathbb{Q}^n$, it plays as important a role as i_p does in the analytic singular homology, which will be defined in §3.1. In our theory of the singular homology and integral, we deal with the spectrum Δ_k^n and $[0,1]_k^n$ in the non-Archimedean world instead of the ordinary standard simplex Δ^n and the ordinary cube $[0,1]^n$ in the Euclidean world. Now we verify analogues between a polytope S and its realisation S_k.

Proposition 1.2.13 (maximum modulus principle)**.** *Let $S \subset \mathbb{R}^n$ be a polytope. Then the Gauss norm $\| \cdot \|_S$ on k_S coincides with the supremum norm* $\sup_{t \in S} i_p(t)$ *on S. Furthermore, since* $\sup_{t \in S} i_p(t) \leq \sup_{t \in S_k} t \leq \| \cdot \|_S$ *as seminorms by the definition of S_k, one has* $\sup_{t \in S_k} t = \| \cdot \|_S$*. In particular if S is thick, one obtains* $\| \cdot \|_S = \sup_{t \in S} \| \cdot \|_t$.

Note that the latter equality $\sup_{t \in S_k} t = \| \cdot \|_S$ follows from the general fact that the equality $\sup_{t \in \mathscr{M}(A)} t = \| \cdot \|$ holods for a uniform k-Banach algebra A by [BER1], Theorem 1.3.1.

Proof. We may and do assume S is thick. Take any $f \in k_S \subset k^{E_{k,n}}$. We prove $\|f\|_S = (\sup_{t \in S} i_p(t))(f) = \sup_{t \in S} |f(t)|$. By the definition of the Gauss norm, we know $\|f\|_S \geq$

$\sup_{t \in S} |f(t)|$. On the other hand, by the definition of the Gauss norm $\| \cdot \|_S$ again, there exists some $x \in E_{k,n}$ such that $\|f\|_S = |f_x| \|x\|_S$ and $|f_y| \|y\|_S \leq |f_x| \|x\|_S$ for any $y \in E_{k,n}$. Since $\|0\|_S = 0 = \sup_{t \in S} |0(t)|$, we may assume $f \neq 0$ and therefore $|f_x| > 0$. By the definition of $\|x\|_S$, there exists some $t \in S$ such that $|x(t)| = \|x\|_S$ because S is a non-empty compact space. One has

$$\sup_{t' \in S} |f(t')| \geq |f(t)| \geq |f_x||x(t)| = |f_x| \|x\|_S = \|f\|_S,$$

and therefore $\sup_{t \in S} |f(t)| = \|f\|_S$. □

Corollary 1.2.14 (Shilov boundary)**.** *For a polytope $S \subset \mathbb{R}^n$, the Shilov boundary ([BER1]) of S_k coincides with the finite discrete subspace consisting of the vertexes of S. In particular if each vertex of S lies in $\mathbb{Z}^n \subset \mathbb{R}^n$, then one has $\|k_S\| = |k| \subset [0, \infty)$.*

Proof. The Shilov boundary of a polytope is the set of its vertexes, and each vertex has an analytic function of the form $\underline{p}(l(t))$ for a system $\underline{p} \in \mathbb{Q}_k^\vee$ of power roots of p and a linear expansion $l = a_1 t_1 + \cdots + a_n t_n + b \in \mathbb{Z}[t_1, \ldots, t_n]$ such that the hyperplain $l(t) = 0$ intersects S only at the vertex. □

Since there are many points in the open subspace $S_k \backslash S$, the correspondence $S \rightsquigarrow S_k$ is not compatible with union, or the convex closure of the union of the underlying topological space. However, if we assume some suitable condition such as the base field k is contained in the completion $\mathbb{C}_p$ of the algebraic closure $\overline{\mathbb{Q}}_p$ of the p-adic number field $\mathbb{Q}_p$, we have good compatibility concerning about union.

Proposition 1.2.15. *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = 1$*, then one has*

$$\mathbb{R}_k^n = \bigcup_{t \in \mathbb{R}^n} t_k$$

for any $n \in \mathbb{N}_+$.

Proof. By the definition of $\mathbb{R}_k^n$, it suffices to show that

$$I_k = \bigcup_{t \in I} t_k$$

for any rectangle $I \subset \mathbb{R}^n$. Take a system $\underline{p} \in \mathbb{Q}_k^\vee$ of power roots of p. Denote by x_i the element $(\underline{p}_i^{(1)} = 1, \ldots, \underline{p}_i^{(i-1)} = 1, \underline{p}_i^{(i)} = \underline{p}, \underline{p}_i^{(i+1)} = 1, \ldots, \underline{p}_i^{(n)} = 1) \in E_{k,n}$ for each $i = 1, \ldots, n$. Present $I = I^{(1)} \times \ldots \times I^{(n)}$ by closed intervals $I^{(i)} \in \mathbb{R}$. Take any point $t' \in I_k$. Since $|\underline{p}_i(t')| \leq \|\underline{p}_i\|_I = |p|^{\min I^{(i)}}$ and $|\underline{p}_i^{-1}(t')| \leq \|\underline{p}_i^{-1}\|_I = |p|^{-\max I^{(i)}}$, one has $|\underline{p}_i(t')| \in [|p|^{\max I^{(i)}}, |p|^{\min I^{(i)}}] = \{|\underline{p}(t^{(i)})| \mid t^{(i)} \in I^{(i)}\} \subset \mathbb{R}$. Now the condition $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = 1$ implies that the restriction $t'|_{E_{k,n}} : E_{k,n} \to [0, \infty)$ is determined by the values $|\underline{p}_1(t')|, \ldots, |\underline{p}_n(t')|$ because of the multiplicatibity of t. One has $t' \leq t := (\log_{|p|} |\underline{p}_1(t')|, \ldots, \log_{|p|} |\underline{p}_n(t')|) \in I \subset I_k$ as a norm because the norm $i_p(t) = \| \cdot \|_t$ is the greatest bounded multiplicative norm in bounded seminorms whose restrictions on $E_{k,n}$ coincide with that of $\| \cdot \|_{t'}$. Therefore $t' \in t_k \subset I_k$. □

Corollary 1.2.16. *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} = 1$*, then one has*

$$S_k = \bigcup_{t \in S} t_k$$

for any thick polytope S*.*

Even though the k-Banch algebra k_S is not an affinoid algebra, we have kind of the universality of an affinoid subdomain when we consider a bounded homomorphism from k_S to an affinoid algebra. The proof of the universality of a general affinoid subdomain is a bit complicated, and we do not whether know it holds or not. Therefore we only see that of a rational domain and a Weierstrass domain as a first step.

Proposition 1.2.17 (universality of a rational domain)**.** *Let* S *be a polytope,* A *a* k*-affinoid algebra,* $V \subset \mathscr{M}(A)$ *a rational domain ([BER1]), and* $\psi \colon A \to k_S$ *a bounded* k*-homomorphism. If the image* $\psi^*(|S_k|) \subset |\mathscr{M}(A)|$ *by the continuous map* $\psi^* \colon |S_k| \to |\mathscr{M}(A)|$ *associated with* $\psi \colon A \to k_S$ *is contained in* $|V| \subset |\mathscr{M}(A)|$*, the homomorphism* $\psi \colon A \to k_S$ *uniquely factors through the canonical homomorphism* $A \to A_V$ *of affinoid algebras.*

To tell the truth, this proposition holds even if we replace k_S to a general unital commutative uniform k-Banach algebra.

Proof. Since k_S is uniform by Proposition 1.1.39, the Gel'fand transform $\Gamma \colon k_S \to B$ is an isometric isomorphism onto its image ([BER1], 1.3.2.), where B is the direct product of the complete valuation fields over k corresponding to points of S_k, and hence we identify k_S as a k-Banach subalgebra of B. Since the image of $t \in S \subset S_k$ in (A) is contained in V, the character $A \to k_S \to k_t$ can be uniquely extended to the character $A_V \to Q(k_t)$, which corresponds to the image of t in V. Therefore one obtains the bounded k-homomorphism $(\Gamma \circ \psi)_V \colon A_V \to B$ which is an extension of the homomorphism $\Gamma \circ \psi \colon A \to k_S \to B$. Now denote by $A_{(V)}$ the localisation $A[f^{-1} \mid |f(t)| \neq 0, t \in V]$, which is dense in A_V by [BER1], 2.2.10. Since $k_S^\times = \{f \in B \mid |f(t)| \neq 0, t \in S_k\}$ by [BER1], 1.2.4, the homomorphism $\psi \colon A \to k_S$ can be uniquely extended to $\psi_{(V)} \colon A_{(V)} \to k_S$ by the universality of the localisation. The homomorphism $(\Gamma \circ \psi)_V \colon A_V \to B$ is the unique extension of $\Gamma \circ \psi_{(V)} \colon A_{(V)} \to k_S \to B$ becuase $A_{(V)}$ is dense in A_V, and it uniquely factors through $\hat{\Gamma} \colon \hat{k_S} \hookrightarrow B$, where $\hat{k_S} \subset B$ is the topological closure of k_S in B. Since k_S is complete and $\Gamma \colon k_S \to B$ is an isometry, k_S is closed in B. It implies that the given homomorphism $\psi \colon A \to k_S$ is uniquely extended to $\psi_V \colon A_V \to k_S$. $\square$

Definition 1.2.18 (reduced Gel'fand transform)**.** *Let* S *be a thick polytope, and set* $C^0(S,k) \equiv \prod_{t \in S} Q(k_t)$*, where* $\prod_{t \in I} Q(k_t)$ *is the direct product as a* k*-Banach algebra. Then one has the canonical projection* $B \to C^0(S,k)$*. The homomorphism* $\Gamma \colon k_S \to B \to C^0(S,k)$ *is an isometric isomorphism onto its image because the norm of* k_S *is the supremum norm on* S*. Call it the reduced Gel'fand transform of* k_S*. We identify* k_S *as a* k*-Banach subalgebra of* $C^0(S,k)$*. One has to take care that the image of* $k_S \backslash \{0\}$ *in* $Q(k_t)$ *is contained in* $Q(k_t)^\times$ *and hence* $k_S^\times \subsetneq \{f \in C^0(S,k) \mid f(t) \neq 0, t \in S\}$*.*

Proposition 1.2.19 (universality of a Weierstrass domain). *Let S be a polytope, A a k-affinoid algebra, $V \subset \mathscr{M}(A)$ a Weierstrass domain ([BER1]), and $\psi \colon A \to k_S$ a bounded k-algebra homomorphism. If the image $\psi^*(|S|) \subset |\mathscr{M}(A)|$ of the underlying polytope $S \subset S_k$ by the continuous map $\psi^* \colon |S_k| \to |\mathscr{M}(A)|$ associated with $\psi \colon A \to k_S$ is contained in $|V| \subset |\mathscr{M}(A)|$, the homomorphism $\psi \colon A \to k_S$ uniquely factors through the canonical homomorphism $A \to A_V$ of affinoid algebras, and hence the image $\psi^*(|S_k|) \subset |\mathscr{M}(A)|$ is contained in $|V| \subset |\mathscr{M}(A)|$.*

Proof. We may and do assume S is thick. Take any $t \in S$, and let $k_S \to Q(k_t)$ be the corresponding character. Since the image of t is contained in V, the character $A \to k_S \to Q(k_t)$ can be uniquely extended to $A_V \to Q(k_t)$, and hence one has obtained the unique extension $(\Gamma \circ \psi)_V \colon A_V \to C^0(S,k)$ of the homomorphism $\Gamma \circ \psi \colon A \to k_S \to C^0(S,k)$. Now the image $(\Gamma \circ \psi)_V(A_V) \subset C^0(S,k)$ is contained in $k_S \subset C^0(S,k)$ because k_S is closed in $C^0(S,k)$ and A is dense in A_V by [BER1], 2.2.10. Therefore one gets the bounded k-algebra homomorphism $(\Gamma \circ \psi)_V \colon A_V \to k_S$. □

Once we have defined "standard simplices" and "cubes" in the non-Archimedean analytic world, the singular homology of a space can be defined by the method of [HAT] and [SER1]. In order to define a morphism from Δ_k^n and $[0,1]_k^n$ to a k-analytic space, we want to define the "atlas" and "covering" of them. We have some analogue of Tate's acyclicity ([BER1]) for the class of the corresponding algebras k_S.

Definition 1.2.20. *Let $S \subset \mathbb{R}^n$ be a thick polytope. Define its interior $\mathrm{Int}(S) \subset S$ as the topological interior in $\mathbb{R}^n$. Then obviously $\mathrm{Int}(S)$ is a dense open subset of S.*

Definition 1.2.21. *Let $S \subset \mathbb{R}^n$ be a polytope. Take a thick polytope $T \subset \mathbb{R}^m$ and an isomorphic integral affine map $a \colon S \to T$. Define the interior $\mathrm{Int}(S) \subset S$ as the pre-image of $\mathrm{Int}(T) \subset T$ by the homeomorphism a. Then obviously $\mathrm{Int}(S)$ is independent of the choice of a thick polytope T and an isomorphic integral affine map a and is a dense open subset of S. The operator Int commutes with an isomorphic affine map.*

Definition 1.2.22. *Let $S \subset T$ be polytopes. The polytope S is said to be a subpolytope of T if $\mathrm{Int}(S)$ is open in T, and then write $S \leq T$.*

The notion "subpolytope" guarantees that the inclusion is equidimensional in the sense of topological manifolds with boundaries.

Lemma 1.2.23. *Let $S \subset T$ be polytopes. If S is thick, then so is T and $S \leq T$.*

Proof. The first implication is trivial because $I(T) \subset I(S) = 0$. When both S and T are thick, then their interiors coincide with their topological interiors in $\mathbb{R}^n$. □

Lemma 1.2.24. *Let $S \leq T$ be polytopes. Then S is thick if and only if T is thick.*

Proof. The direct implication is verifies above. If T is thick, then so is S because the zero locus of a linear expansion $a_1 t_1 + \cdots + a_n t_n + b \in \mathbb{Z}[t_1, \ldots, t_n] \subset \mathrm{C}^0(\mathbb{R}^n, \mathbb{R})$ never contains a small open ball in $\mathbb{R}^n$. □

Lemma 1.2.25. *Let $S \leq T$ be polytopes. Then the restriction map $k_T \to k_S$ is injective.*

Proof. We may and do assume that S and T are thick. Then the injectivity holds by Lemma 1.1.21. □

Now we deal with the analogue of Tate's acyclicity in the non-Archimedean realisations of polytopes. To begin with, we prepare some operators $\wedge$ and $\vee$ which work as the union and the intersection in the category of the G-topology.

Definition 1.2.26. *For polytopes $S, T \subset \mathbb{R}^n$, set*

$$S \wedge T := (Int(S) \cap Int(T))\hat{} \subset \mathbb{R}^n,$$

where $\hat{}$ is the topological closure in $\mathbb{R}^n$, and denote by $S \vee T \subset \mathbb{R}^n$ the convex closure of the topological closure of $Int(S) \cup Int(T) \subset \mathbb{R}^n$. Then each of $S \wedge T$ and $S \vee T$ is a polytope or the empty set. These definitions are stable under the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$. Formally set $\emptyset \wedge S = S \wedge \emptyset := \emptyset$ and $\emptyset \vee S = S \vee \emptyset := S$. The binary operators $\wedge$ and $\vee$ are associative, and are the union and the intersection respectively in the category polytopes and the emptyset. Therefore the notations $S_1 \wedge \cdots \wedge S_m$ and $S_1 \vee \cdots \vee S_m$ make sense. For polytopes $S, S_1, \ldots, S_m$, if $S_1, \ldots, S_m \leq S$ and $Int(S) = Int(S_1) \cup \cdots \cup Int(S_m)$, then write $S = S_1 \veebar \cdots \veebar S_m$. Obvuously $S_1 \veebar \cdots \veebar S_m = S_1 \vee \cdots \vee S_m$.

Lemma 1.2.27 (distributive property)**.** *For polytopes S, T, and $S_1, \ldots, S_m$ satisfying $T \leq S$ and $S = S_1 \veebar \cdots \veebar S_m$, one has*

$$T = (S_1 \wedge T) \veebar \cdots \veebar (S_m \wedge T).$$

Proof.

$$\begin{aligned} & \mathrm{Int}(S_1 \wedge T) \cup \cdots \cup \mathrm{Int}(S_m \wedge T) = (\mathrm{Int}(S_1) \cap \mathrm{Int}(T)) \cup \cdots \cup (\mathrm{Int}(S_m) \cap \mathrm{Int}(T)) \\ = \; & (\mathrm{Int}(S_1) \cup \cdots \cup \mathrm{Int}(S_m)) \cap \mathrm{Int}(T) = \mathrm{Int}(S) \cap \mathrm{Int}(T) = \mathrm{Int}(S) \end{aligned}$$

□

Lemma 1.2.28. *Let $S, S_1, \ldots, S_m$ be polytopes such that $S = S_1 \veebar \cdots \veebar S_m$. Then the direct product*

$$k_S \to \prod_{i=1}^{m} k_{S_i}$$

of the restriction maps $k_S \hookrightarrow k_{S_i}$ is an isometry.

Proof. We may and do assume $S, S_1, \ldots, S_m \subset \mathbb{R}^n$ are thick. Since the topological closure and a locally finite union commute, one has

$$S = (\mathrm{Int}(S))\hat{} = (\mathrm{Int}(S_1) \cup \cdots \cup \mathrm{Int}(S_m))\hat{} = \mathrm{Int}(S_1)\hat{} \cup \cdots \cup \mathrm{Int}(S_m)\hat{} = S_1 \cup \cdots \cup S_m,$$

where $\hat{}$ is the topological closure in $\mathbb{R}^n$, and hence

$$\|\cdot\|_S = \sup_{t \in S} \|\cdot\|_t = \max{}_{i=1}^{m} \sup_{t \in S_i} \|\cdot\|_t = \max{}_{i=1}^{m} \|\cdot\|_{S_i}$$

by the maximum modulus principle, Proposition 1.2.13. Therefore the given homomorphism is an isometry by the definition of the norm of the direct product in the category of k-Banach algebras. $\square$

We have constructed a prototype of the Grothendieck topology of the non-Archimedean realisation S_k of a polytope S by pulling back the structure of the topology of the site (Polytope)$/S$ defined in a natural way. We will strictly define the site S_k, and before that we see that the topology admits the structure sheaf of analytic functions. This is what is called Tate's acyclicity.

Proposition 1.2.29 (Tate's acyclicity)**.** *Let $S, S_1, \ldots, S_m$ be polytopes such that $S = S_1 \veebar \cdots \veebar S_m$. Then the restriction maps $k_S \hookrightarrow k_{S_i}$ induce the admissible exact sequence*

$$0 \to k_S \to \prod_{i=1}^{m} k_{S_i} \to \prod_{i,j=1}^{m} k_{S_i \wedge S_j}$$

of k-Banach algebras.

Proof. We may and do assume S is thick. Then $S_1, \ldots, S_m \leq S$ are also thick. The injectivity of the homomorphism $k_S \to \prod k_{S_i}$ follows from the previous lemma. Once we verify the exactness, the admissibility holds by Banach's open mapping theorem. See [BOU].

Take an arbitrary $(f_i)_i \in \prod k_{S_i}$ such that $f_i|_{S_i \wedge S_j} = f_j\big|_{S_i \wedge S_j} \in k_{S_i \wedge S_j}$ for each $i, j = 1, \ldots, m$. It suffices to show that there exists some $f \in k_S$ such that $f|_{S_i} = f_i \in k_{S_i}$ for any $i = 1, \ldots, m$. Since the polytopes are thick, we regard each ring as a k-vector subspace of $k^{E_{k,n}}$. Now presenting $f_i = \sum f_{i_x} x$, one has $f_{i_x} = f_{j_x}$ for any $i, j = 1, \ldots, m$ by the connectedness of a polytope S, and f_i's determine the unique element $f = \sum f_x x \in k^{E_{k,n}}$. We have only to prove that f is defined on S, i.e. $\lim_{x \in E_{k,n}} |f_x| \|x\|_S = 0$. Suppose $\lim_{x \in E_{k,n}} |f_x| \|x\|_S \neq 0$. Then there exists a counted infinite subset $\{x_0, x_1, \ldots\} \subset E_{k,n}$ such that $|f_{x_l}| \|x_l\|_S \geq M$ for some $M > 0$ and any $l \in \mathbb{N}$. Since $\lim_{x \in E_{k,n}} |f_x| \|x\|_{S_i} = 0$, the subset $F_i := \{x \in E_{k,n} \mid |f_x| \|x\|_{S_i} \geq M\}$ is a finite set for each $i = 1, \ldots, m$. Since $\{x_0, x_1, \ldots\}$ is an infinite set and the set $\cup F_i$ is a finite set, there exists some $l \in \mathbb{N}$ such that $x_l \notin \cup F_i$. For such an integer $l \in \mathbb{N}$, we know $|f_{x_l}| \|x_l\|_{S_i} < M$ for any $i = 1, \ldots, m$ by the definition of $F_1, \ldots, F_m$. It follows that

$$\max{}_{i=1}^{m} |f_{x_l}| \|x_l\|_{S_i} < M \leq \| f_{x_l} \| \|x_l\|_S$$

and this inequality contradicts the fact the homomorphism $k_S \to \prod k_{S_i}$ is an isomorphism. We conclude $\lim_{x \in E_{k,n}} |f_x| \|x\|_S = 0$. We have obtained an analytic function $f \in k_S$, and the restriction $f|_{S_i} \in k_{S_i}$ coincides with f_i by the injectivity of the embeddingd in $k^{E_{k,n}}$. $\square$

Definition 1.2.30. *Denote by* (k*-Alg*) *the category of* k*-algebras whose morphisms are* k*-algebra homomorphisms.*

Corollary 1.2.31. *Let* $S, S_1, \ldots, S_m$ *be polytopes such that* $S = S_1 \veebar \cdots \veebar S_m$*, and* A *a* k*-Banach algebra or a* k*-algebra. The exact sequence of Tate's acyclicity, Proposition 1.2.29, induces the set-theoretical exact sequence*

$$* \to Hom_{(k\text{-}Banach)}(A, k_S) \to \prod_{i=1}^{m} Hom_{(k\text{-}Banach)}(A, k_{S_i}) \rightrightarrows \prod_{i,j=1}^{m} Hom_{(k\text{-}Banach)}(A, k_{S_i \wedge S_j})$$

$$\text{or} \qquad * \to Hom_{(k\text{-}Alg)}(A, k_S) \to \prod_{i=1}^{m} Hom_{(k\text{-}Alg)}(A, k_{S_i}) \rightrightarrows \prod_{i,j=1}^{m} Hom_{(k\text{-}Alg)}(A, k_{S_i \wedge S_j})$$

respectively.

Proof. The exactness is trivial in the case A is k-Banach algebra. In the case A is a k-algebra, the exactness follows from the fact that the forgetful functor (k-Banach) $\to$ (k-Alg) is exact. □

Now we define the structure of S_k as a site. An "open" set is the non-Archimedean realisation of a subpolytope, and an "open covering" is the covering of the interior of the underlying polytope. For the convention of a G-topological space, see [BGR].

Definition 1.2.32. *Let* S *be a polytope. Give* S_k *a G-topology in the following way: An admissible open subset of* S_k *is the empty set* $\emptyset_k \subset S_k$ *or the closed subspace* $T_k \subset S_k$ *corresponding to a subpolytope* $T \leq S$*. Denote by* $\tau_S \subset 2^{S_k}$ *the set of admissible open subsets* $T_k \subset S_k$*. An admissible covering of an admissible open subset* $T_k \subset S_k$ *is a collection of finitely many admissible open subsets* $\{T_{1k}, \ldots, T_{mk} \subset S_k\}$ *such that* $T = T_1 \veebar \ldots \veebar T_m$*. Denote by* $Cov_S(T_k)$ *the set of admissible covering of an admissible open subset* $T_k \subset S_k$*. Note that* $Cov_S(T_k)$ *is independent of the choice of* S_k *containing* T_k *as an admissible open subset. Moreover, endow it the structure presheaf*

$$\begin{array}{rcl} O_S \colon (S_k, \tau_S, Cov_S) & \to & (k\text{-}Banach) \\ T_k & \rightsquigarrow & H^0(T_k, O_S) \coloneqq k_T \\ (i \colon T_k \hookrightarrow T'_k) & \rightsquigarrow & (i^* \colon k_{T'} \to k_T), \end{array}$$

which is a sheaf of k*-Banach algebras by Tate's acyclicity, Proposition 1.2.29. Denote also by* S_k *the G-ringed space* $(S_k, \tau_S, Cov_S, O_S)$ *endowed with the structure sheaf* O_S *of on the G-topology, and call it the spectrum of* k_S *or the non-Archimedean analytic realisation of* S*. When one needs to distinguish the underlying topological space and the underlying G-topological space from the spectrum* S_k*, write* $|S_k|$ *and* $(|S_k|, \tau_S, Cov_S)$ *respectively.*

Definition 1.2.33. *A* k*-affinoid simplex is the spectrum* S_k *for some polytope* S*.*

1.3 General nonsense

Before we define what is a morphism from an affinoid simplex to an analytic space, we prepare a general nonsense: the construction of the amalgamated sum of categories. We will define a morphism from an affinoid simplex to an analytic space as a morphism in the amalgamated sum of the categories (k-Banach) and (k-An) over the cayegory of k-affinoid spaces. This may help readers to understand what we are doing, but we do not assume any facts in this subsection. If a reader does not mind it, he or she might skip whole of this subsection.

Definition 1.3.1. *Let $\mathscr{C}_{-1}, \mathscr{C}_0, \mathscr{C}_1$ be categories endowed with covariant funcors $\iota_0 \colon \mathscr{C}_{-1} \to \mathscr{C}_0$ and $\iota_1 \colon \mathscr{C}_{-1} \to \mathscr{C}_1$. Suppese that the category $\mathscr{C}_{-1}$ is small. For objects $A \in ob\mathscr{C}_0$ and $B \in ob\mathscr{C}_1$, set*

$$Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B) := \bigsqcup_{C \in ob\mathscr{C}_{-1}} Hom_{\mathscr{C}_0}(A, \iota_0(C)) \times Hom_{\mathscr{C}_1}(\iota_1(C), B).$$

For an element $\gamma \in Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B)$, let $C_\gamma \in ob\mathscr{C}_{-1}$ be the unique object such that

$$\gamma \in Hom_{\mathscr{C}_0}(A, \iota_0(C_\gamma)) \times Hom_{\mathscr{C}_1}(\iota_1(C_\gamma), B) \subset Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B),$$

and denot by $\gamma^{(0)} \in Hom_{\mathscr{C}_0}(A, \iota_0(C_\gamma))$ and $\gamma^{(1)} \in Hom_{\mathscr{C}_1}(\iota_1(C_\gamma), B)$ the unique morphisms such that $\gamma = (\gamma^{(0)}, \gamma^{(1)})$.

We will consider the following case: The category $\mathscr{C}_{-1}$ is the category of k-affinoid spaces, which is contravariantly equivalent with the category of k-affinoid algebras. The category $\mathscr{C}_0$ is the opposite category of the full subcategory of (k-Banach) generated by k-affinoid algebras and the essential image of the realisation functor $*\colon$ (Polytope) $\to$ (k-Banach)$\colon S \rightsquigarrow k_S$, or simply the opposite category of (k-Banach). One might extend $\mathscr{C}_0$ to the opposite category of unital associative k-Banach algebras if he or she wants to develop the theory of non-Archimedean non-commutative C^*-algebras or something like that. The category $\mathscr{C}_1$ is the category of k-analytic spaces. The functor ι_0 and ι_1 are the canonical embeddings as a full subcategory, which are trivially fully faithful.

Definition 1.3.2. *For elements $\gamma_0, \gamma_1 \in Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B)$, we write $\gamma_0 \sim \gamma_1$ if there exist an element $\gamma_{0.5} \in Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B)$ and morphisms $\phi_0 \in Hom_{\mathscr{C}_{-1}}(C_{\gamma_{0.5}}, C_{\gamma_0})$ and $\phi_1 \in Hom_{\mathscr{C}_{-1}}(C_{\gamma_{0.5}}, C_{\gamma_1})$ scuh that the diagrams*

$$\begin{array}{ccc} A & \xrightarrow{\gamma_0^{(0)}} & \iota_0(C_{\gamma_0}) \\ \| & & \uparrow{\scriptstyle \iota_0(\phi_0)} \\ A & \xrightarrow{\gamma_{0.5}^{(0)}} & \iota_0(C_{\gamma_{0.5}}) \\ \| & & \downarrow{\scriptstyle \iota_0(\phi_1)} \\ A & \xrightarrow{\gamma_1^{(0)}} & \iota_0(C_{\gamma_1}) \end{array} \qquad \begin{array}{ccc} \iota_1(C_{\gamma_0}) & \xrightarrow{\gamma_0^{(1)}} & B \\ {\scriptstyle \iota_1(\phi_0)}\uparrow & & \| \\ \iota_1(C_{\gamma_{0.5}}) & \xrightarrow{\gamma_{0.5}^{(1)}} & B \\ {\scriptstyle \iota_1(\phi_1)}\downarrow & & \| \\ \iota_1(C_{\gamma_1}) & \xrightarrow{\gamma_1^{(1)}} & B \end{array}$$

commute.

Lemma 1.3.3. *The binary relation $\sim$ is an equvalence relation on $Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B)$.*

Definition 1.3.4. *Set $Hom_{\iota_0,\iota_1}(A, B) := Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B)/\sim$.*

Lemma 1.3.5. *For objects $A, A' \in ob\mathscr{C}_0$ and $B, B' \in ob\mathscr{C}_1$, the composition map*

$$\begin{array}{rcl} \circ\circ \in Hom_{\mathscr{C}_0}(A, A') \times Hom_{\iota_0,\iota_1}(A', \mathscr{C}_{-1}, B') \times Hom_{\mathscr{C}_0}(B', B) & \to & Hom_{\iota_0,\iota_1}(A, \mathscr{C}_{-1}, B) \\ (\phi, (\gamma^{(0)}, \gamma^{(1)}), \psi) & \mapsto & (\gamma^{(0)} \circ \phi, \psi \circ \gamma^{(1)})) \end{array}$$

is invariant under the equivalence relation $\sim$, and hence it induces a map

$$Hom_{\mathscr{C}_0}(A, A') \times Hom_{\iota_0,\iota_1}(A', B') \times Hom_{\mathscr{C}_0}(B', B) \to Hom_{\iota_0,\iota_1}(A, B)$$

Lemma 1.3.6. *For an object $C \in ob\mathscr{C}_{-1}$, consider the canonical maps*

$$\begin{array}{rcl} i_0 \colon Hom_{\mathscr{C}_0}(A, \iota_0(C)) & \to & Hom_{\mathscr{C}_0}(A, \iota_0(C)) \times Hom_{\mathscr{C}_1}(\iota_1(C), \iota_1(C)) \twoheadrightarrow Hom_{\iota_0,\iota_1}(A, \iota_1(C)) \\ \phi & \mapsto & (\phi, id) \mapsto [(\phi, id)] \end{array}$$

and

$$\begin{array}{rcl} i_1 \colon Hom_{\mathscr{C}_1}(\iota_1(C), B) & \to & Hom_{\mathscr{C}_0}(\iota_0(C), \iota_0(C)) \times Hom_{\mathscr{C}_1}(\iota_1(C), B) \twoheadrightarrow Hom_{\iota_0,\iota_1}(\iota_0(C), B) \\ \psi & \mapsto & (id, \psi) \mapsto [(id, \psi)]\,. \end{array}$$

The map i_0 is injective if ι_1 is faithful, and is surjective if ι_1 is full. The map i_1 is injective if ι_0 is faithful, and is surjective if ι_0 is surjective.

Corollary 1.3.7. *If ι_0 and ι_1 are fully faithful, then one has the canonical bijective maps*

$$i_0 \colon Hom_{\mathscr{C}_0}(A, \iota_0(C)) \to Hom_{\iota_0,\iota_1}(A, \iota_1(C))$$

and

$$i_1 \colon Hom_{\mathscr{C}_1}(\iota_1(C), B) \to Hom_{\iota_0,\iota_1}(\iota_0(C), B)$$

for any objects $A \in ob\mathscr{C}_0$, $B \in ob\mathscr{C}_1$, and $C \in ob\mathscr{C}_{-1}$. In particular, for objects $C_0, C_1 \in ob\mathscr{C}_{-1}$, one obtains the canonical bijective maps

$$i_0 \circ \iota_0 \colon Hom_{\mathscr{C}_{-1}}(C_0, C_1) \to Hom_{\mathscr{C}_0}(\iota_0(C_0), \iota_0(C_1)) \to Hom_{\iota_0,\iota_1}(\iota_0(C_0), \iota_1(C_1))$$

and

$$i_1 \circ \iota_1 \colon Hom_{\mathscr{C}_{-1}}(C_0, C_1) \to Hom_{\mathscr{C}_1}(\iota_1(C_0), \iota_1(C_1)) \to Hom_{\iota_0,\iota_1}(\iota_0(C), \iota_1(C_1)),$$

and they coincide with each other.

Corollary 1.3.8. *If ι_0 and ι_1 are fully faithful, then for objects $C_0, C_1 \in ob\mathscr{C}_{-1}$, one obtains the canonical identification*

$$Hom_{\iota_0,\iota_1}(\iota_0(C_0), \iota_1(C_1)) = Hom_{\iota_1,\iota_0}(\iota_1(C_0), \iota_0(C_1)).$$

Definition 1.3.9. *Suppose ι_0 and ι_1 are fully faithful. Define the category $\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1$ setting data in the following way: Set*

$$ob(\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1) := ob(\mathscr{C}_0) \sqcup ob(\mathscr{C}_1),$$

in general or

$$ob(\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1) := ob(\mathscr{C}_0) \cup_{ob(\mathscr{C}_{-1})} ob(\mathscr{C}_1)$$

if the class enriched with the categories admits the amalgamated sum $(\cdot)\cup_{(\cdot)}(\cdot)$. Whichever definition one chooses, the equivalence class of the constrcuted category does not change. For objects $C, C' \in ob(\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1)$, set

$$Hom_{\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1}(C, C') := Hom_{\mathscr{C}_0}(C, C')$$

if $C, C' \in ob(\mathscr{C}_0)$,

$$Hom_{\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1}(C, C') := Hom_{\mathscr{C}_1}(C, C')$$

if $C, C' \in ob(\mathscr{C}_1)$,

$$Hom_{\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1}(C, C') := Hom_{\iota_0, \iota_1}(C, C')$$

if $C \in ob(\mathscr{C}_0)$ and $C' \in ob(\mathscr{C}_1)$, and

$$Hom_{\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1}(C, C') := Hom_{\iota_1, \iota_0}(C, C')$$

if $C \in ob(\mathscr{C}_1)$ and $C' \in ob(\mathscr{C}_0)$. This definition is well-defined by the properties above. For obejects $C, C', C'' \in ob(\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1)$, one has the canonical composition

$$Hom_{\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1}(C, C') \times Hom_{\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1}(C', C'') := Hom_{\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1}(C, C'')$$

in a natural way.

Lemma 1.3.10. *The canonical embeddings $\mathscr{C}_0 \hookrightarrow \mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1$ and $\mathscr{C}_1 \hookrightarrow \mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1$ are fully faithful and the diagram*

$$\begin{array}{ccc} \mathscr{C}_0 & \longrightarrow & \mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1 \\ \uparrow & & \uparrow \\ \mathscr{C}_{-1} & \longrightarrow & \mathscr{C}_1 \end{array}$$

commutes up to a natural equivalence.

Lemma 1.3.11. *The category $\mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1$ satisfies the universality of the amalgamated sum of categories. Namely, for a category $\mathscr{C}$ and functors $\mathscr{F}_0 \colon \mathscr{C}_0 \to \mathscr{C}$ and $\mathscr{F}_1 \colon \mathscr{C}_1 \to \mathscr{C}$, if the compositions $\mathscr{F}_0 \circ \iota_0 \colon \mathscr{C}_{-1} \to \mathscr{C}_0 \to \mathscr{C}$ and $\mathscr{F}_1 \circ \iota_1 \colon \mathscr{C}_{-1} \to \mathscr{C}_1 \to \mathscr{C}$ are naturally equivalent, then there is a unique functor $\mathscr{F}_0 \cup \mathscr{F}_1 \colon \mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1 \to \mathscr{C}$ up to natural equivalence such that the compositions $\mathscr{C}_0 \hookrightarrow \mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1 \to \mathscr{C}$ and $\mathscr{C}_1 \hookrightarrow \mathscr{C}_0 \cup_{\mathscr{C}_{-1}} \mathscr{C}_1 \to \mathscr{C}$ are naturally equivalent with $\mathscr{F}_0$ and $\mathscr{F}_1$ respectively.*

We have finished the general nonsense.

1.4 Analytic path from an affinoid simplex

We define a morphism from an affinoid simplex to an analytic space. Similar with the position of an affine scheme in the theory of schemes and with that of an affinoid space in the theory of analytic spaces, one has the analogue of the adjoint property between the Berkovich's spectrum functor $\mathscr{M} \colon A \rightsquigarrow \mathscr{M}(A)$ and the global section functor $\mathrm{H}^0(\cdot, \mathbb{G}_a) \colon X \rightsquigarrow \mathrm{H}^0(X, O_X)$. Throughout this paper, we mention an analytic space in the sense of [BER2] but not [BER1]. A k-affinoid algebra is a k-Banach algebra admitting an admissible epimorphism from a general Tate algebra $k\{r^{-1}T\}$, which is not necessarily a classical Tate algebra $k\{T\}$.

Definition 1.4.1. *Denote by* $(k\text{-}An)$ *the category of k-analytic spaces in the sense of [BER2]. For a k-analytic space* X*, denote by* $|X|$ *the underlying topological space. For a morphism* $\gamma \colon X \to Y$*, denote by* $\gamma^\sharp \colon |X| \to |Y|$ *the underlying continuous map, and by* $\gamma^a = (\gamma^a_{V,W})_{(V,W)}$ *the associated system of k-algebra homomorphisms* $\gamma^a_{V,W} \colon H^0(W, O_Y) \to H^0(V, O_X)$ *for analytic domains* $V \subset X$ *and* $W \subset Y$ *satisfying* $\gamma^\sharp(|V|) \subset |W|$.

In particular for a bounded k-algebra homomorphism $\phi \colon A \to B$ between k-affinoid algebras, associate the continuous map $\mathscr{M}(\phi)^\sharp = \phi^* \colon |\mathscr{M}(B)| \to |\mathscr{M}(A)|$. In order to avoid flooding of notations, we write $\mathscr{M}(\phi)^\sharp$ instead of ϕ^* in this subsection.

Definition 1.4.2. *Denote by* $\mathscr{A}_k$ *the family of all k-Banach algebras of the form* $k\{T_1, \ldots, T_n\}/I$ *for an integer* $n \in \mathbb{N}$ *and a proper ideal* $I \subsetneq k\{T_1, \ldots, T_n\}$*. Note that* $\mathscr{A}_k$ *is not a proper class in the sense of Von Neumann-Bernays-Gödel set theory, and represents all isomorphic classes of k-affinoid algebras.*

Definition 1.4.3. *Let* S_k *be a k-affinoid simplex and* X *a k-analytic space. Set*

$$Hom(S, \mathscr{A}_k, X) := \bigsqcup_{A \in \mathscr{A}_k} Hom_{(k\text{-}Banach)}(A, k_S) \times Hom_{(k\text{-}An)}(\mathscr{M}(A), X).$$

For an element $\gamma \in Hom(S, \mathscr{A}_k, X)$*, let* $A_\gamma \in \mathscr{A}_k$ *be the unique k-affinoid algebra such that*

$$\gamma \in Hom_{(k\text{-}Banach)}(A_\gamma, k_S) \times Hom_{(k\text{-}An)}(\mathscr{M}(A_\gamma), X).$$

Denote by $\gamma^{(0)} \colon k_S \to A_\gamma$ *and* $\gamma^{(1)} \colon \mathscr{M}(A_\gamma) \to X$ *the unique bounded k-algebra homomorphism and the unique morphism such that*

$$\gamma = (\gamma^{(0)}, \gamma^{(1)}) \in Hom_{(k\text{-}Banach)}(A_\gamma, k_S) \times Hom_{(k\text{-}An)}(\mathscr{M}(A_\gamma), X).$$

Definition 1.4.4. *For two elements* $\gamma_0, \gamma_1 \in Hom(S, \mathscr{A}_k, X)$*, we write* $\gamma_0 \sim \gamma_1$ *if there exist an element* $\gamma_{0.5} \in Hom(S, \mathscr{A}_k, X)$ *and bounded k-algebra homomorphisms* $\phi_{0.5,0} \colon A_{\gamma_0} \to$

$A_{\gamma_{0.5}}$ and $\phi_{0.5,1} \colon A_{\gamma_1} \to A_{\gamma_{0.5}}$ such that the diagrams

$$
\begin{tikzcd}
k_S \arrow[d, equal] & A_{\gamma_0} \arrow[l, "\gamma_0^{(0)}"'] \arrow[d, "\phi_{0.5,0}"] \\
k_S \arrow[d, equal] & A_{\gamma_{0.5}} \arrow[l, "\gamma_{0.5}^{(0)}"'] \\
k_S & A_{\gamma_1} \arrow[l, "\gamma_1^{(0)}"'] \arrow[u, "\phi_{0.5,1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M}(A_{\gamma_0}) \arrow[r, "\gamma_0^{(1)}"] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_{0.5}}) \arrow[u, "\mathscr{M}(\phi_{0.5,0})"] \arrow[r, "\gamma_{0.5}^{(1)}"] \arrow[d, "\mathscr{M}(\phi_{0.5,1})"'] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_1}) \arrow[r, "\gamma_1^{(1)}"] & X
\end{tikzcd}
$$

commute.

Lemma 1.4.5. *The binary relation $\sim$ on Hom$(S, \mathscr{A}_k, X)$ is an equivalence relation.*

Proof. The reflexivity and the semmetry are trivial. We verify the transitivity. Take three elements $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Hom}(S, \mathscr{A}_k, X)$ satisfying $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$. There are elements $\gamma_{0.5}, \gamma_{1.5} \in \mathrm{Hom}(S, \mathscr{A}_k, X)$ such that the diagrams

$$
\begin{tikzcd}
k_S \arrow[d, equal] & A_{\gamma_0} \arrow[l, "\gamma_0^{(0)}"'] \arrow[d, "\phi_{0.5,0}"] \\
k_S \arrow[d, equal] & A_{\gamma_{0.5}} \arrow[l, "\gamma_{0.5}^{(0)}"'] \\
k_S & A_{\gamma_1}, \arrow[l, "\gamma_1^{(0)}"'] \arrow[u, "\phi_{0.5,1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k_S \arrow[d, equal] & A_{\gamma_1} \arrow[l, "\gamma_1^{(0)}"'] \arrow[d, "\phi_{1.5,1}"] \\
k_S \arrow[d, equal] & A_{\gamma_{1.5}} \arrow[l, "\gamma_{1.5}^{(0)}"'] \\
k_S & A_{\gamma_2}, \arrow[l, "\gamma_2^{(0)}"'] \arrow[u, "\phi_{1.5,2}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathscr{M}(A_{\gamma_0}) \arrow[r, "\gamma_0^{(1)}"] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_{0.5}}) \arrow[u, "\mathscr{M}(\phi_{0.5,0})"] \arrow[r, "\gamma_{0.5}^{(1)}"] \arrow[d, "\mathscr{M}(\phi_{0.5,1})"'] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_1}) \arrow[r, "\gamma_1^{(1)}"] & X,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M}(A_{\gamma_1}) \arrow[r, "\gamma_1^{(1)}"] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_{1.5}}) \arrow[u, "\mathscr{M}(\phi_{1.5,1})"] \arrow[r, "\gamma_{1.5}^{(1)}"] \arrow[d, "\mathscr{M}(\phi_{1.5,2})"'] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_2}) \arrow[r, "\gamma_2^{(1)}"] & X
\end{tikzcd}
$$

commute. Denote by $\phi_{0.5,1.5,1} \colon A_{\gamma_1} \to A_{\gamma_{0.5}} \hat{\otimes}_{A_{\gamma_1}} A_{\gamma_{1.5}}$ the canonical bounded k-algebra

homomorphism $(\phi_{0.5,0} \otimes \mathrm{id}) \circ \phi_{1.5,0} = (\mathrm{id} \otimes \phi_{1.5,0}) \circ \phi_{0.5,0}$. Then the diagrams

$$
\begin{tikzcd}
k_S \arrow[d, equal] & A_{\gamma_0} \arrow[l, "\gamma_0^{(0)}"'] \arrow[d, "(\mathrm{id}\otimes\phi_{1.5,0})\circ\phi_{0.5,0}"] \\
k_S \arrow[d, equal] & A_{\gamma_{0.5}} \hat{\otimes}_{A_{\gamma_1}} A_{\gamma_{1.5}} \arrow[l, "\gamma_{0.5,1.5}^{(0)}"'] \\
k_S & A_{\gamma_2} \arrow[l, "\gamma_1^{(0)}"'] \arrow[u, "(\phi_{0.5,1}\otimes\mathrm{id})\circ\phi_{1.5,0}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M}(A_{\gamma_0}) \arrow[r, "\gamma_0^{(1)}"] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_{0.5}} \hat{\otimes}_{A_{\gamma_1}} A_{\gamma_{1.5}}) \arrow[u, "\mathscr{M}(\phi_{0.5,0})\circ\mathscr{M}(\mathrm{id}\otimes\phi_{1.5,0})"'] \arrow[d, "\mathscr{M}(\phi_{1.5,0})\circ\mathscr{M}(\mathrm{id}\otimes\phi_{0.5,0})"] \arrow[r, "\gamma_1^{(1)}\circ\mathscr{M}(\phi_{0.5,1.5,1})"] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_2}) \arrow[r, "\gamma_2^{(1)}"] & X
\end{tikzcd}
$$

commute, and hence $\gamma_0 \sim \gamma_2$. □

Definition 1.4.6. *Set Hom*$(S, X) \coloneqq$ *Hom*$(S, \mathscr{A}_k, X)/\sim$. *Call an element* $\gamma \in$ *Hom*(S, X) *a morphism, and write* $\gamma \colon S \to X$.

Definition 1.4.7. *When* S *is the cube* $[0,1]^n$*, the standerd simplex*

$$\Delta^n \coloneqq \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, t_0 + \cdots + t_n = 1, 0 \leq t_i \leq 1, {}^\forall i = 0, \ldots, n \right\},$$

or something like them, then call a morphism from S *an analytic path.*

Defining morphisms in this way, we do not have to introduce the notion of an atlas of an affinoid simplex. If we used an atlas of an affinoid simplex relating with its G-topology, we should consider the special G-topology, and the corresponding equivalence relation would become too loose. It is due to the facts that the pull-back of an affinoid domain $V \subset \mathscr{M}(A)$ by the continuous map $|S_k| \to |\mathscr{M}(A)|$ associated with a bounded k-algebra homomorphism is not an admissible open subset of S_k in general, and that an affinoid domain which is not rational does not have the universality for a morphism from an affinoid simplex. We should also consider the gluing problem of a morphism, too. We do not have Tate's acyclicity for a G-topology finer than the G-topology given by subpolytopes, and hence the gluing is pretty troublesome.

Now we see that this definition of a morphism $S \to X$ is calculated as $\mathrm{Hom}_{(k\text{-Banach})}(A, k_S)$ when X is an affinoid space $\mathscr{M}(A)$. In other words, there is the adjoint property between the global section functor

$$\begin{aligned} \mathrm{H}^0(\cdot, \mathbb{G}_a) \colon X &\rightsquigarrow \mathrm{H}^0(X, O_X) \\ (\phi \colon X \to Y) &\rightsquigarrow (\mathrm{H}^0(\cdot, \mathbb{G}_a)(\phi) = \phi^{\mathrm{a}}_{X,Y} \colon \mathrm{H}^0(Y, O_Y) \to \mathrm{H}^0(X, O_X)) \end{aligned}$$

and Berkovich's spectrum functor

$$\begin{aligned} \mathscr{M} \colon A &\rightsquigarrow \mathscr{M}(A) \\ (\phi \colon A \to B) &\rightsquigarrow (\mathscr{M}(\phi) = \phi^* \colon \mathscr{M}(B) \to \mathscr{M}(A)) \end{aligned}$$

with respect to a morphism from an affinoid simplex to an affinoid space.

Lemma 1.4.8. *For an element $\gamma \in Hom(S, \mathscr{A}_k, X)$, set*

$$H^0(\cdot, \mathbb{G}_a)(\gamma) \coloneqq \gamma^{(0)} \circ H^0(\cdot, \mathbb{G}_a)(\gamma^{(1)})\colon H^0(X, O_X) \xrightarrow{H^0(-,\mathbb{G}_a)(\gamma^{(1)})} A_\gamma \xrightarrow{\gamma^{(0)}} k_S\,.$$

This correspondence

$$\begin{array}{rcl} H^0(\cdot, \mathbb{G}_a)\colon Hom(S, \mathscr{A}_k, X) & \to & Hom_{(k\text{-}Alg)}(H^0(X, O_X), k_S) \\ \gamma & \to & H^0(\cdot, \mathbb{G}_a)(\gamma) \end{array}$$

is invariant under the equivalence relation $\sim$, and hence it determines a set-theoretical map

$$\begin{array}{rcl} H^0(\cdot, \mathbb{G}_a)\colon Hom(S, X) & \to & Hom_{(k\text{-}Alg)}(H^0(X, O_X), k_S) \\ \gamma & \to & H^0(\cdot, \mathbb{G}_a)(\gamma). \end{array}$$

Proof. Take two elements $\gamma_0, \gamma_1 \in \mathrm{Hom}(S, \mathscr{A}_k, X)$ satisfying $\gamma_0 \sim \gamma_1$. There is an element $\gamma_{0.5} \in \mathrm{Hom}(S, \mathscr{A}_k, X)$ such that the diagrams

$$\begin{tikzcd} k_S \arrow[d, equal] & A_{\gamma_0} \arrow[l, "\gamma_0^{(0)}"'] \arrow[d, "\phi_{0.5,0}"] \\ k_S \arrow[d, equal] & A_{\gamma_{0.5}} \arrow[l, "\gamma_{0.5}^{(0)}"'] \\ k_S & A_{\gamma_1} \arrow[l, "\gamma_1^{(0)}"'] \arrow[u, "\phi_{0.5,1}"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{M}(A_{\gamma_0}) \arrow[r, "\gamma_0^{(1)}"] & X \arrow[d, equal] \\ \mathscr{M}(A_{\gamma_{0.5}}) \arrow[u, "\mathscr{M}(\phi_{0.5,0})"] \arrow[r, "\gamma_{0.5}^{(1)}"] \arrow[d, "\mathscr{M}(\phi_{0.5,1})"'] & X \arrow[d, equal] \\ \mathscr{M}(A_{\gamma_1}) \arrow[r, "\gamma_1^{(1)}"] & X \end{tikzcd}$$

commute. It follows the diagram

$$\begin{tikzcd} k_S \arrow[d, equal] & A_{\gamma_0} \arrow[l, "\gamma_0^{(0)}"'] \arrow[d, "\phi_{0.5,0}"'] & \mathrm{H}^0(X, O_X) \arrow[l, "\mathrm{H}^0(-,\mathbb{G}_a)(\gamma_0^{(1)})"'] \arrow[d, equal] \\ k_S \arrow[d, equal] & A_{\gamma_{0.5}} \arrow[l, "\gamma_{0.5}^{(0)}"'] & \mathrm{H}^0(X, O_X) \arrow[l, "\mathrm{H}^0(-,\mathbb{G}_a)(\gamma_{0.5}^{(1)})"'] \arrow[d, equal] \\ k_S & A_{\gamma_1} \arrow[l, "\gamma_1^{(0)}"'] \arrow[u, "\phi_{0.5,1}"] & \mathrm{H}^0(X, O_X) \arrow[l, "\mathrm{H}^0(-,\mathbb{G}_a)(\gamma_1^{(1)})"'] \end{tikzcd}$$

commutes, and $\mathrm{H}^0(\cdot, \mathbb{G}_a)\colon \mathrm{Hom}(S, \mathscr{A}_k, X) \to \mathrm{Hom}_{(k\text{-}\mathrm{Alg})}(\mathrm{H}^0(X, O_X), k_S)$ is invariant under the equivalence relation $\sim$. □

Similarly one also has the underlying continuous map of a morphism.

Lemma 1.4.9. *For an element $\gamma \in Hom(S, \mathscr{A}_k, X)$, set*

$$\gamma^\sharp := \gamma^{(1)\sharp} \circ \mathscr{M}(\gamma^{(0)})^\sharp \colon |S_k| \xrightarrow{\mathscr{M}(\gamma^{(0)})^\sharp} |\mathscr{M}(A_\gamma)| \xrightarrow{\gamma^{(1)\sharp}} |X|.$$

This correspondence

$$\begin{aligned} \sharp \colon Hom(S, \mathscr{A}_k, X) &\to Hom_{(Top)}(|S_k|, |X|) \\ \gamma &\to \gamma^\sharp \end{aligned}$$

is invariant under the equivalent relation $\sim$, and hence it determines a set-theoretical map

$$\begin{aligned} \sharp \colon Hom(S, X) &\to Hom_{(Top)}(|S_k|, |X|) \\ \gamma &\to \gamma^\sharp. \end{aligned}$$

Call the image $\gamma^\sharp \colon |S_k| \to |X|$ of a morphism $\gamma \colon S \to X$ the underlying continuous map of γ.

Proof. It can be easily verified in a similar way of the proof pf the previous lemma. □

Proposition 1.4.10 (adjoint property)**.** *Let S_k be a k-affinoid simplex and $\mathscr{M}(A)$ a k-affinoid space. The canonical map*

$$H^0(\cdot, \mathbb{G}_a) \colon Hom(S, \mathscr{M}(A)) \to Hom_{(k\text{-}Alg)}(A, k_S)$$

defined above induces a set-theoretical bijection

$$H^0(\cdot, \mathbb{G}_a) \colon Hom(S, \mathscr{M}(A)) \to Hom_{(k\text{-}Banach)}(A, k_S).$$

Proof. First, we verify that $\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma) \in \mathrm{Hom}_{(k\text{-Alg})}(A, k_S)$ is bounded for any morphism $\gamma \colon S \to \mathscr{M}(A)$. Take a representative $\underline{\gamma} \in \mathrm{Hom}(S, \mathscr{A}_k, \mathscr{M}(A))$. Then by the construction of $\mathrm{H}^0(\gamma)$, one has

$$\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma) = \underline{\gamma}^{(0)} \circ \mathrm{H}^0(\cdot, \mathbb{G}_a)(\underline{\gamma}^{(1)}).$$

Since the correspondence

$$\mathscr{M} \colon \mathrm{Hom}_{(k\text{-Banach})}(A, A_{\underline{\gamma}}) \to \mathrm{Hom}_{(k\text{-An})}(\mathscr{M}(A_{\underline{\gamma}}), \mathscr{M}(A))$$

is a bijective map whose inverse map is the correspondence

$$\mathrm{H}^0(\cdot, \mathbb{G}_a) \colon \mathrm{Hom}_{(k\text{-An})}(\mathscr{M}(A_{\underline{\gamma}}), \mathscr{M}(A)) \to \mathrm{Hom}_{(k\text{-Banach})}(A, A_{\underline{\gamma}}),$$

the k-algebra homomorphism $\mathrm{H}^0(\cdot, \mathbb{G}_a)(\underline{\gamma}^{(1)}) \colon A \to A_{\underline{\gamma}}$ is bounded. Therefore $\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma) \in \mathrm{Hom}_{(k\text{-Banach})}(A, k_S)$.

Secondly, we prove that the induced map

$$\mathrm{H}^0(\cdot, \mathbb{G}_a) \colon \mathrm{Hom}(S, \mathscr{M}(A)) \to \mathrm{Hom}_{(k\text{-Banach})}(A, k_S)$$

is bijective. We construct the inverse map

$$\mathscr{M} \colon \mathrm{Hom}_{(k\text{-Banach})}(A, k_S) \to \mathrm{Hom}(S, \mathscr{M}(A)).$$

Take an admissible epimorphism $k\{T_1, \ldots, T_n\} \twoheadrightarrow A$, and let $I \subsetneq k\{T_1, \ldots, T_n\}$ be the kernel. Denote by ι the isomorphism $k\{T_1, \ldots, T_n\}/I \to A$. For a bounded k-algebra homomorphism $\phi \colon A \to k_S$, denote by $\mathscr{M}(\phi) \in \mathrm{Hom}(S, \mathscr{M}(A))$ the equivalent class of the element $(\phi \circ \iota^{-1}, \mathscr{M}(\iota)) \in \mathrm{Hom}_{(k\text{-Banach})}(k\{T_1, \ldots, T_n\}/I, k_S) \times \mathrm{Hom}_{(k\text{-An})}(\mathscr{M}(k\{T_1, \ldots, T_n\}/I), \mathscr{M}(A)$. Then the morphism $\mathscr{M}(\phi) \colon S \to \mathscr{M}(A)$ is independent of the choice of the admissible epimorphism $k\{T_1, \ldots, T_n\} \twoheadrightarrow A$. Indeed, take two admissible epimorphisms $\iota_0 \colon k\{T_1, \ldots, T_{n_0}\} \twoheadrightarrow A$ and $\iota_1 \colon k\{T_1, \ldots, T_{n_1}\} \twoheadrightarrow A$, and let $I_0 \subsetneq k\{T_1, \ldots, T_{n_0}\}$ and $I_1 \subsetneq k\{T_1, \ldots, T_{n_1}\}$ be their kernels. Denote by ι_0 and ι_1 the induced isomorphisms $k\{T_1, \ldots, T_{n_0}\}/I_0 \to A$ and $k\{T_1, \ldots, T_{n_1}\}/I_1 \to A$ respectively. Then the diagrams

$$\begin{array}{ccc}
k_S & \xleftarrow{\phi \circ \iota_0^{-1}} & k\{T_1, \ldots, T_{n_0}\}/I_0 \\
\| & & \| \\
k_S & \xleftarrow{\phi \circ \iota_0^{-1}} & k\{T_1, \ldots, T_{n_0}\}/I_0 \\
\| & & \uparrow {\scriptstyle \iota_0^{-1} \circ \iota_1} \\
k_S & \xleftarrow{\phi \circ \iota_1^{-1}} & k\{T_1, \ldots, T_{n_1}\}/I_1
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{M}(k\{T_1, \ldots, T_{n_0}\}/I_0) & \xrightarrow{\mathscr{M}(\iota_0)} & \mathscr{M}(A) \\
\| & & \| \\
\mathscr{M}(k\{T_1, \ldots, T_{n_0}\}/I_0) & \xrightarrow{\mathscr{M}(\iota_0)} & \mathscr{M}(A) \\
{\scriptstyle \mathscr{M}(\iota_0^{-1} \circ \iota_1)} \downarrow & & \| \\
\mathscr{M}(k\{T_1, \ldots, T_{n_1}\}/I_1) & \xrightarrow{\mathscr{M}(\iota_1)} & \mathscr{M}(A)
\end{array}$$

commute. Therefore $(\phi \circ \iota_0^{-1}, \mathscr{M}(\iota_0)) \sim (\phi \circ \iota_1^{-1}, \mathscr{M}(\iota_1))$.

Finally we show that the correspondence

$$\begin{array}{rcl}
\mathscr{M} \colon \mathrm{Hom}_{(k\text{-Banach})}(A, k_S) & \to & \mathrm{Hom}(S, \mathscr{M}(A)) \\
\phi & \mapsto & \mathscr{M}(\phi).
\end{array}$$

is the inverse map of $\mathrm{H}^0(\cdot, \mathbb{G}_a)$. Fix an isomorphism $\iota \colon k\{T_1, \ldots, T_n\}/I \to A$. Take a bounded k-algebra homomorphism $\phi \colon A \to k_S$. Then one has

$$\begin{aligned}
\mathrm{H}^0(\cdot, \mathbb{G}_a)(\mathscr{M}(\phi)) &= \mathrm{H}^0(\cdot, \mathbb{G}_a)(\phi \circ \iota^{-1}, \mathscr{M}(\iota)) = (\phi \circ \iota^{-1}) \circ \mathrm{H}^0(\cdot, \mathbb{G}_a)(\mathscr{M}(\iota)) \\
&= (\phi \circ \iota^{-1}) \circ \iota = \phi,
\end{aligned}$$

and hence $\mathrm{H}^0(\cdot, \mathbb{G}_a) \circ \mathscr{M} = \mathrm{id}$. On the other hand, take a morphism $\gamma \colon S \to \mathscr{M}(A)$ and

fix a representative $\underline{\gamma} \in \mathrm{Hom}(S, \mathscr{A}_k, \mathscr{M}(A))$. Then the diagrams

$$
\begin{tikzcd}
k_S \arrow[d, equal] & A_{\underline{\gamma}} \arrow[l, "\underline{\gamma}^{(0)}"'] \arrow[d, equal] \\
k_S \arrow[d, equal] & A_{\underline{\gamma}} \arrow[l, "\underline{\gamma}^{(0)}"'] \\
k_S & k\{T_1, \ldots, T_n\}/I \arrow[l, "\underline{\gamma}^{(0)} \circ \iota^{-1}"'] \arrow[u, "\mathrm{H}^0(-, \mathbb{G}_a)(\underline{\gamma}^{(1)}) \circ \iota"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M}(A_{\underline{\gamma}}) \arrow[r, "\underline{\gamma}^{(1)}"] \arrow[d, equal] & \mathscr{M}(A) \arrow[d, equal] \\
\mathscr{M}(A_{\underline{\gamma}}) \arrow[r, "\underline{\gamma}^{(1)}"] \arrow[d, "\mathscr{M}(\iota) \circ \underline{\gamma}^{(1)}"'] & \mathscr{M}(A) \arrow[d, equal] \\
\mathscr{M}(k\{T_1, \ldots, T_n\}/I) \arrow[r, "\mathscr{M}(\iota^{-1})"] & \mathscr{M}(A)
\end{tikzcd}
$$

commute and hence one obtains

$$
\begin{aligned}
\mathscr{M}(\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma)) &= \mathscr{M}(\underline{\gamma}^{(1)} \circ \mathrm{H}^0(\cdot, \mathbb{G}_a)(\underline{\gamma}^{(0)})) \\
&= [(\mathrm{H}^0(\cdot, \mathbb{G}_a)(\underline{\gamma}^{(0)})) \circ \mathscr{M}(\underline{\gamma}^{(1)}) \circ \iota^{-1}, \mathscr{M}(\iota))] \\
&= [\underline{\gamma}] = \gamma.
\end{aligned}
$$

Therefore $\mathscr{M} \circ \mathrm{H}^0(\cdot, \mathbb{G}_a) = \mathrm{id}$. □

We identify $\mathrm{Hom}(S, \mathscr{M}(A)) = \mathrm{Hom}_{(k\text{-Banach})}(A, k_S)$ by this canonical bijective map. This map is functorial in the following sense:

Proposition 1.4.11. *Let S_k and T_k be k-affinoid simplices, and X and Y k-analytic spaces. The maps*

$$
\begin{aligned}
Hom_{(Polytope)}(S, T) \times Hom_{(k\text{-}Banach)}(A, k_T) &\to Hom_{(k\text{-}Banach)}(A, k_S) \\
(a, \phi) &\to \phi \circ a^*
\end{aligned}
$$

and

$$
\begin{aligned}
Hom_{(k\text{-}An)}(\mathscr{M}(A), X) \times Hom_{(k\text{-}An)}(X, Y) &\to Hom_{(k\text{-}An)}(\mathscr{M}(A), Y) \\
(\chi, \psi) &\to \psi \circ \chi
\end{aligned}
$$

for a k-affinoid algebra $A \in \mathscr{A}_k$ induce the composition map

$$
\begin{aligned}
Hom_{(Polytope)}(S, T) \times Hom(T, X) \times Hom_{(k\text{-}An)}(X, Y) &\to Hom(S, Y) \\
(a, \phi, \psi) &\to \psi \circ \phi \circ a
\end{aligned}
$$

Remark that each of the image $a'(U') \subset T$ of a polytope $U' \subset S$ and the preimage $a'^{-1}(U'') \subset S$ of a polytope $U'' \subset T$ by an integral affine map $a' \colon S \to T$ is a polytope or the empty set, that an integral affine map preserves a covering, and that the restriction $a'|_{U'}^{U''} \colon U' \to U''$ on polytopes $U' \subset S$ and $U'' \subset T$ containing $a'(U')$ is also an integral affine map.

Proof. The composition maps are obviously invariant under the equivalence relation $\sim$. $\square$

Proposition 1.4.12. *Let S_k and T_k be k-affinoid simplices, and $\mathscr{M}(A)$ and $\mathscr{M}(B)$ k-affinoid spaces. The diagram*

$$\begin{array}{ccc} Hom_{(Polytope)}(S,T) \times Hom(T, \mathscr{M}(A)) \times Hom_{(k\text{-}An)}(\mathscr{M}(A), \mathscr{M}(B)) & \xrightarrow{\circ \times \circ} & Hom(S, \mathscr{M}(B)) \\ \Big\downarrow{\scriptstyle * \times H^0(-,\mathbb{G}_a) \times H^0(-,\mathbb{G}_a)} & & \Big\downarrow{\scriptstyle H^0(-,\mathbb{G}_a)} \\ Hom_{(k\text{-}Banach)}(k_T, k_S) \times Hom_{(k\text{-}Banach)}(A, k_T) \times Hom_{(k\text{-}Banach)}(B, A) & \xrightarrow{\circ \times \circ} & Hom_{(k\text{-}Banach)}(B, k_S) \end{array}$$

commutes.

Proof. Trivial by the construction of each correspondence. $\square$

By the definition of a morphism, one knows a morphism from an affinoid simplex always factors through an affinoid space.

Proposition 1.4.13. *The map*

$$\circ \colon \bigsqcup_{A \in \mathscr{A}_k} Hom(S, \mathscr{M}(A)) \times Hom_{(k\text{-}An)}(\mathscr{M}(A), X) \to Hom(S, X)$$

induced by the composition

$$\begin{array}{rcl} \circ \colon Hom(S, \mathscr{M}(A)) \times Hom_{(k\text{-}An)}(\mathscr{M}(A), X) & \to & Hom(S, X) \\ (\gamma, \phi) & \mapsto & \phi \circ \gamma \end{array}$$

is surjective.

Proof. Take a morphism $\gamma \colon S \to X$, and fix a representative $\underline{\gamma} \in \mathrm{Hom}(S, \mathscr{A}_k, X)$ of γ. There is an isomorphism $\iota \colon A \to A_{\underline{\gamma}}$ for some $A \in \mathscr{A}_k$. Set

$$\underline{\gamma'} := (\underline{\gamma}^{(0)} \circ \iota, \mathrm{id}) \in \mathrm{Hom}_{(k\text{-Banach})}(A, k_S) \times \mathrm{Hom}_{(k\text{-An})}(\mathscr{M}(A), \mathscr{M}(A)) \subset \mathrm{Hom}(S, \mathscr{A}_k, \mathscr{M}(A))$$

and

$$\phi := \underline{\gamma}^{(1)} \circ \mathscr{M}(\iota^{-1}) \colon \mathscr{M}(A) \to X.$$

Then obviously one has $\gamma = \phi \circ [\underline{\gamma'}]$. $\square$

We did not make use of the G-topology of a k-affinoid simplex in the definition of a morphism, but one also has a sheaf-theoretic structure of a morphism.

Definition 1.4.14. *For a morphism $\gamma \colon S \to X$, let $\Lambda(\gamma)$ be the collection of pairs (T, V) of a subpolytope $T \leq S$ and an analytic domain $V \subset X$ such that there exist a representative $\underline{\gamma} \in Hom(S, \mathscr{A}_k, X)$ of γ and a rational domain $W \subset \mathscr{M}(A_{\underline{\gamma}})$ such that $\mathscr{M}(\underline{\gamma}^{(0)})^\sharp(|T_k|) \subset |W| \subset |\mathscr{M}(A_{\underline{\gamma}})|$ and $\underline{\gamma}^{(1)\sharp}(|W|) \subset |V| \subset |X|$.*

Proposition 1.4.15. *For a morphism $\gamma\colon S \to X$, there is a canonical functorial system $(\gamma^a_{T,V})_{(T,V)\in\Lambda(\gamma)}$ of k-algebra homomorphisms $\gamma^a_{T,V}\colon H^0(V,O_X) \to k_T$ satisfying the following properties:*

(i) for pairs $(T,V),(T',V') \in \Lambda(\gamma)$ satisfying $T' \leq T$ and $V \subset V'$, the diagram

$$\begin{array}{ccc} k_T & \xleftarrow{\gamma^a_{T,V}} & H^0(V,O_X) \\ \downarrow & & \uparrow \\ k_{T'} & \xleftarrow{\gamma^a_{T',V'}} & H^0(V',O_X) \end{array}$$

commutes.

(ii) for pairs $(T,V),(T',V') \in \Lambda(\gamma)$ satisfying $T' \leq T$ and $V' \subset V$, the diagram

$$\begin{array}{ccc} k_T & \xleftarrow{\gamma^a_{T,V}} & H^0(V,O_X) \\ \downarrow & & \downarrow \\ k_{T'} & \xleftarrow{\gamma^a_{T',V'}} & H^0(V',O_X) \end{array}$$

commutes.

(iii) for a pair $(T,V) \in \Lambda(\gamma)$, the diagram

$$\begin{array}{ccc} |T_k| & \xrightarrow{\mathscr{M}(\gamma^a_{T,V})^\sharp} & |V| \\ \downarrow & & \downarrow \\ |S_k| & \xrightarrow{\gamma^\sharp} & |X| \end{array}$$

commutes.

(iv) for a pair $(T,V) \in \Lambda(\gamma)$ satisfying that $V \subset X$ is a special domain, the k-algebra homomorphim $\gamma^a_{T,V}\colon A_V = H^0(V,O_X) \to k_T$ is bounded.

Proof. We just construct the system, and omit the proof of functoriality and other properties. For a pair $(T,V) \in \Lambda(\gamma)$, take a representative $\underline{\gamma} \in \mathrm{Hom}(S,\mathscr{A}_k,X)$ and a rational domain $W \subset \mathscr{M}(A_{\underline{\gamma}})$ such that $\mathscr{M}(\underline{\gamma}^{(0)})^\sharp(|T_k|) \subset |W| \subset |\mathscr{M}(A_{\underline{\gamma}})|$ and $\underline{\gamma}^{(1)\sharp}(|W|) \subset |V| \subset |X|$. Since the image $\mathscr{M}(\underline{\gamma}^{(0)})^\sharp(|T_k|) \subset |W| \subset |\mathscr{M}(A_{\underline{\gamma}})|$, the bounded k-algebra homomorpshim

$$A_{\underline{\gamma}} \xrightarrow{\underline{\gamma}^{(0)}} k_S \to k_T$$

is uniquely extended to a bounded k-algebra homomorpshim $\underline{\gamma}^{(0)}_{W,T}\colon A_W \to k_T$ by the universality of a rational domain, Proposition 1.2.17. Set

$$\gamma^{\mathrm{a}}_{T,V} \coloneqq \underline{\gamma}^{(0)}_{W,T} \circ (\underline{\gamma}^{(1)})^{\mathrm{a}}_{V,W}\colon \mathrm{H}^0(V,O_X) \xrightarrow{(\underline{\gamma}^{(1)})^{\mathrm{a}}_{V,W}} A_W \xrightarrow{\underline{\gamma}^{(0)}_{W,T}} k_T.$$

This k-algebra homomorphism $\gamma^{\mathrm{a}}_{T,V}$ is independent of the choice of $\underline{\gamma}$ and W, and determines a canonical functorial system $(\gamma^{\mathrm{a}}_{T,V})_{(T,V)\in\Lambda(\gamma)}$. □

Moreover we introduce the notion of the ground field extension of a morphism $S \to X$. In the case X is affinoid, the ground field extension functor is compatible with the identification $\mathrm{H}^0(\cdot, \mathbb{G}_a)$.

Proposition 1.4.16 (ground field extension)**.** *Let K/k be an extension of complete non-Archimedean fields, S_k a k-affinoid simplex, and X a k-analytic space. The ground field extensions*

$$\begin{array}{rl} & (\cdot)_K \colon Hom_{(k\text{-}Banach)}(A, k_S) \to Hom_{(K\text{-}Banach)}(A_K, K_S) \\ \textit{and} & (\cdot)_K \colon Hom_{(k\text{-}An)}(\mathscr{M}(A), X) \to Hom_{(K\text{-}An)}(\mathscr{M}(A)_K, X_K) \end{array}$$

for a k-affinoid algebra $A \in \mathscr{A}_k$ induce the correspondence

$$(\cdot)_K \colon Hom(S, X) \to Hom(S, X_K).$$

Also call it the ground field extention.

Proof. The equivalence relation $\sim$ is obviously stable under the ground field extension. □

Recall that we constructed the Galois action on k_S. The composition of the Galois action $G_k \to \mathrm{Aut}_{(k\text{-Banach})}(k_S)$ and the composition map $\circ\colon \mathrm{Hom}_{(k\text{-Banach})}(A, k_S) \times \mathrm{Aut}_{(k\text{-Banach})}(k_S)$ for a k-affinoid algebra A induces a Galois action $\mathrm{Hom}(S, X) \times G_k \to \mathrm{Hom}(S, X)$. This Galois action is significant because it induces a Galois action on the analytic homologies which will be defined in §3.1. The analytic homologies are thus Galois representations.

Proposition 1.4.17 (Galois action)**.** *The Galois action*

$$\begin{array}{rcl} Hom_{(k\text{-}Banach)}(A, k_S) \times G_k & \to & Hom_{(k\text{-}Banach)}(A, k_S) \\ (\phi, g) & \mapsto & g \circ \phi \end{array}$$

for a k-affinoid algebra $A \in \mathscr{A}_k$ induce the Galois action

$$\begin{array}{rcl} Hom(S, \mathscr{A}_k, X) \times G_k & \to & Hom(S, \mathscr{A}_k, X) \\ (\gamma, g) & \mapsto & \gamma \circ g \coloneqq (g^{-1} \circ \gamma^{(0)}, \gamma^{(1)}) \end{array}$$

which preserves the equivalence relation $\sim$. Therefore it gives an well-defined Galois action

$$\begin{array}{rcl} Hom(S, X) \times G_k & \to & Hom(S, X) \\ (\gamma, g) & \mapsto & \gamma \circ g. \end{array}$$

Proof. The equivalence relation $\sim$ is obviously stable under the Galois action. □

Finally we see the universalities of the fibre product, the direct limit, a rational domain, and a Weierstrass domain.

Proposition 1.4.18 (universality of the fibre product)**.** *Let S_k be a k-affinoid simplex, X, Y k-analytic spaces, and $\mathscr{M}(A)$ a k-affinoid space. For any morphisms $\phi \colon X \to \mathscr{M}(A)$ and $\psi \colon Y \to \mathscr{M}(A)$, one has the canonical functorial set-theoretical bijective map*

$$Hom(S, X) \times_{Hom(S, \mathscr{M}(A))} Hom(S, Y) \to Hom(S, X \times_{\mathscr{M}(A)} Y).$$

Be caureful about the fact we do not have the same universality for the fibre product $X \times_Y Z$ for arbitrary analytic spaces X, Y, Z. It is bacause the fibra product of two affinoid spaces over an analytic space is not an affinoid space. See the proof.

Proof. To begin with, one constructs a set-theoretical map

$$\mathrm{Hom}(S, X) \times_{\mathrm{Hom}(S, \mathscr{M}(A))} \mathrm{Hom}(S, Y) \to \mathrm{Hom}(S, X \times_{\mathscr{M}(A)} Y).$$

Take morphisms $\gamma_0 \colon S \to X$ and $\gamma_1 \colon S \to Y$ satisfying $\phi \circ \gamma_0 = \psi \circ \gamma_1 \colon S \to \mathscr{M}(A)$. Fix representatives $\underline{\gamma_0} \in \mathrm{Hom}(S, \mathscr{A}_k, X)$ and $\underline{\gamma_1} \in \mathrm{Hom}(S, \mathscr{A}_k, Y)$ of γ_0 and γ_1 respectively. Set

$$\phi' := \mathrm{H}^0(\cdot, \mathbb{G}_a)(\phi \circ \underline{\gamma_0}^{(1)}) \colon A \to A_{\underline{\gamma_0}},$$

$$\psi' := \mathrm{H}^0(\cdot, \mathbb{G}_a)(\psi \circ \underline{\gamma_1}^{(1)}) \colon A \to A_{\underline{\gamma_1}},$$

$$A_{\underline{\gamma}_{0,1}} := A_{\underline{\gamma_0}} \hat{\otimes}_A A_{\underline{\gamma_1}},$$

$$\phi'' := \underline{\gamma_0}^{(0)} \circ \mathscr{M}(\mathrm{id} \otimes \psi') \colon \mathscr{M}(A_{\underline{\gamma}_{0,1}}) \to X,$$

and

$$\psi'' := \underline{\gamma_1}^{(0)} \circ \mathscr{M}(\phi' \otimes \mathrm{id}) \colon \mathscr{M}(A_{\underline{\gamma}_{0,1}}) \to Y,$$

Since $\phi \circ \gamma_0 = \psi \circ \gamma_1$, the diagrams

$$\begin{array}{ccc} k_S & \xleftarrow{\underline{\gamma_1}^{(0)}} & A_{\underline{\gamma_1}} \\ {\scriptstyle \underline{\gamma_0}^{(0)}}\uparrow & & \uparrow{\scriptstyle \psi'} \\ A_{\underline{\gamma_0}} & \xleftarrow{\phi'} & A. \end{array} \qquad \begin{array}{ccc} \mathscr{M}(A_{\underline{\gamma_{0,1}}}) & \xrightarrow{\psi''} & Y \\ {\scriptstyle \phi''}\downarrow & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{\phi} & \mathscr{M}(A). \end{array}$$

commute. Set

$$\underline{\gamma_{0,1}}^{(0)} := \underline{\gamma_0}^{(0)} \otimes \underline{\gamma_1}^{(0)} \colon A_{\underline{\gamma_{0,1}}} \to k_S$$

and

$$\underline{\gamma_{0,1}}^{(1)} := \phi'' \times \psi'' \colon \mathscr{M}(A_{\underline{\gamma_{0,1}}}) \to X \times_{\mathscr{M}(A)} Y.$$

Let $\gamma_{0,1} \colon S \to X \times_{\mathscr{M}(A)} Y$ be the equivalence class of $(\underline{\gamma_{0,1}}^{(0)}, \underline{\gamma_{0,1}}^{(1)}) \in \mathrm{Hom}(S, \mathscr{A}_k, X \times_{\mathscr{M}(A)} Y)$. Then $\gamma_{0,1}$ is independent of the choice of the representatives $\underline{\gamma_0}$ and $\underline{\gamma_1}$, and one obtained the map

$$\begin{array}{rcl} \mathrm{Hom}(S, X) \times_{\mathrm{Hom}(S, \mathscr{M}(A))} \mathrm{Hom}(S, Y) & \to & \mathrm{Hom}(S, X \times_{\mathscr{M}(A)} Y) \\ (\gamma_0, \gamma_1) & \to & \gamma_{0,1}. \end{array}$$

It is bijective and has the inverse map

$$\begin{array}{rcl} \mathrm{Hom}(S, X \times_{\mathscr{M}(A)} Y) & \to & \mathrm{Hom}(S, X) \times_{\mathrm{Hom}(S, \mathscr{M}(A))} \mathrm{Hom}(S, Y) \\ \gamma & \to & ((\mathrm{id} \times \psi) \circ \gamma, (\phi \times \mathrm{id}) \circ \phi). \end{array}$$

□

Proposition 1.4.19 (universality of the direct limit)**.** *Let I be a directed set, S_k a k-affinoid simplex, X a k-analytic space, and $W \colon I \to (k\text{-}An) \colon i \rightsquigarrow W_i$ a direct system of k-analytic domains of X whose transitive map $W_i \to W_j$ is the inclusion $W_i \hookrightarrow W_j \hookrightarrow X$ for any $i \leq j \in I$. Suppose W converges to X and determines a topological covering of X, i.e.*

$$X = \varinjlim_{i \in I} W_i, \quad |X| = \bigcup_{i \in I} |Int(W_i / X)|,$$

where $Int(W_i/X)$ is the relative interior, [BER1], which coincides with the topological interior in X in this case by [BER1], 3.1.3/(i). Then the canonical set-theoretical map

$$\varinjlim_{i \in I} Hom(S, W_i) \to Hom(S, X)$$

induced by the composition $Hom(S, W_i) \to Hom(S, X)$ of the embedding $\iota_i \colon W_i \hookrightarrow X$ is bijective.

Proof. The composition $\mathrm{Hom}(S, W_i) \to \mathrm{Hom}(S, X)$ of the embedding $\iota_i \colon W_i \hookrightarrow X$ is injective for any $i \in I$. Indeed, take two morphisms $\gamma_0, \gamma_1 \colon S \to W_i$ and suppose $\iota \circ \gamma_0 = \iota \circ \gamma_1 \colon S \to X$. Fix representatives $\underline{\gamma_0}, \underline{\gamma_1} \in \mathrm{Hom}(S, \mathscr{A}_k, W_i)$ of γ_0 and γ_1 respectively. Since

$$(\underline{\gamma_0}^{(0)}, \iota \circ \underline{\gamma_0}^{(1)}) \sim (\underline{\gamma_1}^{(0)}, \iota \circ \underline{\gamma_1}^{(1)}),$$

there exist some element $\underline{\gamma_{0.5}} \in \mathrm{Hom}(S, \mathscr{A}_k, X)$ and bounded k-algebra homomorphisms

$\phi_{0.5,0}\colon A_{\gamma_0}$ and $\phi_{0.5,1}\colon A_{\gamma_1}$ such that the diagrams

$$\begin{tikzcd}
k_S \arrow[d, equal] & A_{\underline{\gamma_0}} \arrow[l, "\underline{\gamma_0}^{(0)}"'] \arrow[d, "\phi_{0.5,0}"] \\
k_S \arrow[d, equal] & A_{\underline{\gamma_{0.5}}} \arrow[l, "\underline{\gamma_{0.5}}^{(0)}"'] \\
k_S & A_{\underline{\gamma_1}} \arrow[l, "\underline{\gamma_1}^{(0)}"'] \arrow[u, "\phi_{0.5,1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M}(A_{\underline{\gamma_0}}) \arrow[r, "\iota\circ\underline{\gamma_0}^{(1)}"] & X \arrow[d, equal] \\
\mathscr{M}(A_{\underline{\gamma_{0.5}}}) \arrow[r, "\underline{\gamma_{0.5}}^{(1)}"] \arrow[u, "\mathscr{M}(\phi_{0.5,0})"] \arrow[d, "\mathscr{M}(\phi_{0.5,1})"'] & X \arrow[d, equal] \\
\mathscr{M}(A_{\underline{\gamma_1}}) \arrow[r, "\iota\circ\underline{\gamma_1}^{(1)}"] & X
\end{tikzcd}$$

commute. The latter commutative diagram gives the commutative diagram

$$\begin{tikzcd}
\mathscr{M}(A_{\underline{\gamma_0}}) \arrow[r, "\underline{\gamma_0}^{(1)}"] & W_i \arrow[d, equal] \\
\mathscr{M}(A_{\underline{\gamma_{0.5}}}) \arrow[r, "\exists!"] \arrow[u, "\mathscr{M}(\phi_{0.5,0})"] \arrow[d, "\mathscr{M}(\phi_{0.5,1})"'] & W_i \arrow[d, equal] \\
\mathscr{M}(A_{\underline{\gamma_1}}) \arrow[r, "\underline{\gamma_1}^{(1)}"] & W_i
\end{tikzcd}$$

by the universality of an analytic domain $W_i \subset X$, and it follows that

$$\underline{\gamma_0} \sim \underline{\gamma_1}.$$

Furthermore, the map

$$\varinjlim_{i\in I} \mathrm{Hom}(S, \mathscr{A}_k, W_i) \to \mathrm{Hom}(S, \mathscr{A}_k, X)$$

is surjective. Indeed, take an element $\gamma \in \mathrm{Hom}(S, \mathscr{A}_k, X)$. Since an affinoid space is compact, there exists some $i \in I$ such that $\gamma^{(1)\sharp}(|\mathscr{M}(A_\gamma)|) \subset |W_i| \subset |X|$. By the universality of the analytic domain $W_i \subset X$, there is the unique morphism $\gamma_1^{(1)}\colon \mathscr{M}(A_\gamma) \to W_i$ such that$\iota \circ \gamma_1^{(1)} = \gamma^{(1)}$. Set

$$\gamma_1 \coloneqq (\gamma^{(0)}, \gamma_1^{(1)}) \in \mathrm{Hom}_{(k\text{-Banach})}(A_\gamma, k_S) \times \mathrm{Hom}_{(k\text{-An})}(A_\gamma, W_i),$$

and then one has $\iota \circ \gamma_1 = \gamma$. Thus

$$\varinjlim_{i\in I} \mathrm{Hom}(S, \mathscr{A}_k, W_i) \to \mathrm{Hom}(S, \mathscr{A}_k, X)$$

is surjective. $\square$

Corollary 1.4.20 (universality of the fibre product)**.** *Let S_k be a k-affinoid simplex, and X, Y, Z k-analytic spaces. Suppose Z admits a direct system $W \colon I \to (k\text{-}An) \colon i \rightsquigarrow W_i$ of k-affinoid domains of Z satisfying the conditions that the transitive map $W_i \to W_j$ is the inclusion $W_i \hookrightarrow W_j \hookrightarrow Z$ for any $i \leq j \in I$, and that W converges to Z and determines a topological covering of X. For any morphisms $\phi \colon X \to Z$ and $\psi \colon Y \to Z$, one has the canonical functorial set-theoretical bijective map*

$$Hom(S, X) \times_{Hom(S,Z)} Hom(S, Y) \to Hom(S, X \times_Z Y).$$

Proof. Straightforward from the previous two propositions. □

Corollary 1.4.21. *Let S_k be a k-affinoid simplex, $\{S_{1k}, \ldots, S_{mk}\} \in Cov_S(S_k)$, and X k-analytic space. Suppose X admits a direct system $W \colon I \to (k\text{-}An) \colon i \rightsquigarrow W_i$ of k-affinoid domains of X satisfying the conditions that the transitive map $W_i \to W_j$ is the inclusion $W_i \hookrightarrow W_j \hookrightarrow X$ for any $i \leq j \in I$, and that W converges to X and determines a topological covering of X. Then the sequence*

$$* \to Hom(S, X) \to \prod_{i=1}^{m} Hom(S_i, X) \rightrightarrows \prod_{i,j=1}^{m} Hom(S_i \wedge S_j, X)$$

is exact.

Note that we need the assumption that X is a direct limit of affinoid domains because of the restriction that a morphism always factors through an affinoid space. The fibre product of finitely many affinoid spaces over a non-affinoid non-separated analytic space is not an affinoid space in general, but the fibre product of finitely many affinoid spaces over an analytic space which is a direct limit of affinoid domains is an affinoid space.

Proof. Straightforward from the corollary of Tate's acyclicity, Proposition 1.2.29, and the universality of a direct limit, Proposition 1.4.19. □

Proposition 1.4.22 (universality of a rational domain)**.** *Let S_k be a k-affinoid simplex, A a k-affinoid algebra, $V \subset \mathscr{M}(A)$ a rational domain, and $\gamma \colon S \to \mathscr{M}(A)$ a morphism. If the image $\gamma^\sharp(|S_k|) \subset |\mathscr{M}(A)|$ is contained in $|V| \subset |\mathscr{M}(A)|$, the morphism $\gamma \colon S \to \mathscr{M}(A)$ uniquely factors through the embedding $V \hookrightarrow \mathscr{M}(A)$.*

Proposition 1.4.23 (universality of a Weierstrass domain)**.** *Let S_k be a k-affinoid simplex, A a k-affinoid algebra, $V \subset \mathscr{M}(A)$ a Weierstrass domain, and $\gamma \colon S \to \mathscr{M}(A)$ a morphism. If the image $\gamma^\sharp(|S|) \subset |\mathscr{M}(A)|$ of the underlying polytope $S \subset S_k$ is contained in $|V| \subset |\mathscr{M}(A)|$, the morphism $\gamma \colon S \to \mathscr{M}(A)$ uniquely factors through the embedding $V \hookrightarrow \mathscr{M}(A)$.*

Proof. These propositions directly follow from the functoriality of the identification $\mathrm{H}^0(\cdot, \mathbb{G}_a)$, Proposition 1.4.12; the universality of a rational domain, Proposition 1.2.17; and the universality of a Weierstrass domain, Proposition 1.2.19. □

Though we do not have the universality of an analytic domain or an affinoid domain for a morphism from an affinoid simplex, the universalities of a rational domain and a Weierstrass domain of an affinoid space are easily extended. We introduce the extended notion of a rational domain and a Weierstrass domain of an affinoid space.

Definition 1.4.24. *Let X be a k-analytic space. An analytic domain $U \subset X$ is said to be a rational domain (or a Weierstrass domain) if for any morphism $\phi \colon \mathscr{M}(A) \to X$ from a k-affinoid space the pull-back $\phi^{-1}(U) \subset \mathscr{M}(A)$ is a rational domain (resp. a Weierstrass domain) of the affinoid space $\mathscr{M}(A)$.*

For example, an analytic subset $U \subset X$ of the form

$$U := \left\{ t \in X \,\middle|\, |f_i(t)| \leq |f_0(t)|, {}^\forall i = 1, \ldots, m \right\} \subset X$$

for some global sections $f_0, \ldots, f_m \in \mathrm{H}^0(X, O_X)$ satisfying $(f_0(t), \ldots, f_m(t)) \neq (0, \ldots, 0)$ for any $t \in X$ is a rational domain, and an analytic subset $U \subset X$ of the form

$$U := \left\{ t \in X \,\middle|\, |f_i(t)| \leq d_i, {}^\forall i = 1, \ldots, m \right\} \subset X$$

for some global sections $f_1, \ldots, f_m \in \mathrm{H}^0(X, O_X)$ and parametres $d_1, \ldots, d_m \in (0, \infty)$ is a Weierstrass domain. Moreover, the notions of a rational domain and a Weierstrass domain are compatible with the original ones when we consider an affinoid space. One has the universality for them.

Proposition 1.4.25 (universality of a rational domain)**.** *Let S_k be a k-affinoid simplex, X a k-analytic space, $V \subset X$ a rational domain, and $\gamma \colon S \to X$ a morphism. If the image $\gamma^\sharp(|S_k|) \subset |X|$ is contained in $|V| \subset |X|$, the morphism $\gamma \colon S \to X$ uniquely factors through the embedding $V \hookrightarrow X$.*

Proposition 1.4.26 (universality of a Weierstrass domain)**.** *Let S_k be a k-affinoid simplex, X a k-analytic space, $V \subset X$ a Weierstrass domain, and $\gamma \colon S \to X$ a morphism. If the image $\gamma^\sharp(|S|) \subset |X|$ of the underlying polytope $S \subset S_k$ is contained in $|V| \subset |X|$, the morphism $\gamma \colon S \to X$ uniquely factors through the embedding $V \hookrightarrow X$.*

Proof. We only verify the first proposition. The universality of a Weierstrass domain is also verified in the same way. Fix a representative $\underline{\gamma} \in \mathrm{Hom}(S, \mathscr{A}_k, X)$. Since the pull-back $V := (\underline{\gamma}^{(1)})^{-1}(U) \subset \mathscr{M}(A_{\underline{\gamma}})$ is a rational domain containing the image $\mathscr{M}(\underline{\gamma}^{(0)})(|S_k|) \subset |\mathscr{M}(A_{\underline{\gamma}})|$, the bounded k-algebra homomorphism $\underline{\gamma}^{(0)} \colon A_{\underline{\gamma}} \to k_S$ is uniquely extended to a bounded k-algebra homomorphism $\underline{\gamma}^{(0)}_V \colon A_V \to k_S$ by the universality of a rational domain, Proposition 1.2.17. Let $\gamma|^U \colon S \to U$ be the equivalence class of an element

$$(\underline{\gamma}^{(0)}_V, \underline{\gamma}^{(1)}|^U_V) \in \mathrm{Hom}_{(k\text{-Banach})}(A_V, k_S) \times \mathrm{Hom}_{(k\text{-An})}(V, U) \subset \mathrm{Hom}(S, \mathscr{M}_k, U),$$

and then obviously $\iota \circ \gamma|^U = \gamma \colon S \to X$, where ι is the inclusion $U \hookrightarrow X$. □

Relating with a Weierstrass domain, we introduce the notion of a Stein space in the sense of [BER1]. A Stein space has good properties concerning about the cohomology of coherent sheaves and the de Rham cohomology as is dealt with in [BER4].

Definition 1.4.27. *A k-analytic space X is said to be Stein if it admits an increasing filtration $X_0 \subset X_1 \subset \cdots \subset \cup X_i = X$ of X by affinoid domains X_i such that $X_i \subset Int(X_{i+1}/X)$ for each $i \in \mathbb{N}$ and X_i is a Weierstrass domain of X_{i+1} for each $i \in \mathbb{N}$, where $Int(X_{i+1}/X)$ is the relative interior. Call the sequence $X_0 \subset X_1 \subset \cdots \subset \cup X_i = X$ a Weierstrass filtration of X.*

Any relatively compact subspace is contained in a sufficiently large X_i. Each affinoid domain $X_i \subset X$ is a Weierstrass domain, and an analytic domain $U \subset X$ is a Weierstrass domain if and only if it is a Weierstrass domain of the affinoid space X_i for some $i \in \mathbb{N}$ for any Weierstrass filtration $(X_i)_{i \in \mathbb{N}}$.

Definition 1.4.28. *Let X be a k-analytic soace, and $U \subset X$ an analytic domain. If U is Stein, call it a Stein subspace. Moreover a Stein subspace $U \subset X$ is said to be a Stein domain if U admits a Weierstrass filtration consisting of Weierstrass domains of X.*

In other words, a Stein domain is an analytic domain which is the direct limit of Weierstrass domains along a directed set $\mathbb{N}$. In particular, a Weierstrass domain is a Stein domain. Note that a Weierstrass domain is locally closed in a strong sense: the pull-back by a morphism from an affinoid space is always closed. On the other hand, a Stein space can be an open subspace. For example, For example, an analytic subset $U \subset X$ of the form

$$U := \left\{ t \in X \;\middle|\; |f_i(t)| < d_i, {}^{\forall} i = 1, \ldots, m \right\} \subset X$$

for some global sections $f_1, \ldots, f_m \in \mathrm{H}^0(X, O_X)$ and parametres $d_1, \ldots, d_m \in (0, \infty)$ is a Stein domain.

At the end of this subsection, we define the notion of a great domain, which is an analytic domain possessing the same universality as that of a Weierstrass domain. In particular a Weierstrass domain is a great domain.

Definition 1.4.29. *Let X be a k-analytic space. An analytic domain $U \subset X$ is said to be a great domain of X if U has the following universality. For any k-affinoid simplex S_k and any morphism $\gamma \colon S \to X$, if the image $\gamma^\sharp(|S|) \subset |X|$ of the underlying polytope $S \subset S_k$ is contained in $|U| \subset |X|$, the morphism $\gamma \colon S \to X$ uniquely factors through the inclusion $U \hookrightarrow X$. If $U_K \subset X_K$ is a great domain for any extention K/k of complete non-Archimedean fields, then we say $U \subset X$ is a universally great domain.*

Proposition 1.4.30. *A Stein domain is a universal great domain.*

Proof. The ground field extension of a Stein domain is again a Stein domain, and hence it suffices to verify that a Stein domain is a great domain. Let X be a k-analytic space and $U \subset X$ a Stein domain. Take a Weierstrass filtration $U_0 \subset U_1 \subset \cdots \subset \cup U_i = U$ consisting

of Weierstrass domains $U_i \subset X$. For any k-affinoid simplex S_k and any morphism $\gamma\colon S \to X$, suppose the image $\gamma^\sharp(|S|) \subset |X|$ of the underlying polytope $S \subset S_k$ is contained in $|U| \subset |X|$. Since $|S|$ is compact, there exists some $i \in \mathbb{N}$ such that $|S| \subset |U_i| \subset |U| \subset |X|$. Since $U_i \subset X$ is a Weierstrass domain, the morphism $\gamma\colon S \to X$ uniquely factors through the inclusion $U_i \hookrightarrow X$, and hence one obtains a morphism $S \to U_i \hookrightarrow U$. $\square$

1.5 Analytic calculations

In this subsection, we calculate the set $\mathrm{Hom}(S, X)$ for significant examples of analytic spaces X, making use of Proposition 1.4.10 and Proposition 1.4.19.

Lemma 1.5.1. *The set-theoretical map*

$$\begin{array}{rcl} Hom_{(k\text{-}Banach)}(k\{d^{-1}T\}, k_S) & \to & \{f \in (k_S)^m \mid \|f\|_S \leq d\} \\ \phi & \mapsto & (\phi(T_1), \dots, \phi(T_m)) \end{array}$$

induces the canonical bijective map

$$Hom(S, D_k^m(d)) \to \{\, f \in (k_S)^m \mid \|f\|_S \leq d \,\},$$

where $d = (d_1, \dots, d_m) \in (0, \infty)^m$ and $D_k^m(d)$ is the polydisc $\mathscr{M}(k\{d^{-1}T\})$.

Proof. We have only to show that the map

$$\begin{array}{rcl} \mathrm{Hom}_{k\text{-Banach}}(\mathbb{D}_k^1(d_1), k_S) \to \{\, f \in k_S \mid \|f\|_S \leq d_1 \,\} & & \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is bijective, because

$$k\{d^{-1}T\} \cong_k k\{d_1^{-1}T_1\}\hat{\otimes}_k \cdots \hat{\otimes}_k k\{d_m^{-1}T_m\}.$$

It follows from the general fact of a bounded k-algebra homomorphism between k-Banach algebras. For a uniform k-Banach algebra A, it is well-known that the map

$$\begin{array}{rcl} \mathrm{Hom}_{(k\text{-Banach})}(k\{d_1^{-1}T_1\}, A) & \to & A \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is injective and its image coincides with the subset $\{f \in A \mid \|f\| \leq d_1\} \subset A$. $\square$

Corollary 1.5.2. *The bijective map above induces the canonical bijective maps*

$$Hom(S, \mathring{D}_k^m(d)) \cong \{\, f \in (k_S)^m \mid \|f\|_S < d \,\}$$

and

$$Hom(S, \mathbb{A}_k^m) \cong (k_S)^m,$$

where $\mathring{D}_k^m(d)$ is the open disc $\cup_{d'<d}\mathscr{M}(k\{d'^{-1}T\})$ and $\mathbb{A}_k^m$ is the affine space $\cup\mathscr{M}(k\{d^{-1}T\})$.

Proof. Trivial because

$$\mathring{\mathrm{D}}_k^m(d) = \varinjlim_{d'<d} \mathscr{M}(k\{d^{-1}T\}).$$

and

$$\mathbb{A}_k^m = \varinjlim_{d\geq 0} \mathscr{M}(k\{d^{-1}T\}).$$

$\square$

It may be one of the most important property of the affine space $\mathbb{A}_k^m$. They are regarded as group objects in the class of k-analytic spaces and analytic rectangles over k, and in particular the affine line $\mathbb{G}_a = \mathbb{A}_k^1$ gives us the structure sheaf O_S with respect to any suitable G-topology of S. Next, we calculate the unit circle $\mathrm{A}_k^1(1,1)$.

Definition 1.5.3. *Let A be a seminormed k-algebra. An element $a \in A$ is said to be power-bounded if the set $\{a^n \mid n \in \mathbb{N}\} \subset A$ is bounded, and is said to be topologically nilpotent if* $\lim_{n\to\infty} a^n = 0$ *or equivalently* $\lim_{n\to\infty} \|a^n\| = 0$. *Denote by $A^\circ \subset A$ the subset of power-bounded elements, and by $A^{\circ\circ}$ the subset of topologically nilpotent elements. In particular k° is the valuation ring of k, and $k^{\circ\circ}$ is the maximal ideal of k°. In general, A° is a closed k°-subalgebra of A and $A^{\circ\circ}$ is an open ideal of A°. In particular if A is uniform, then A° (or $A^{\circ\circ}$) coincides with the closed (resp. open) unit ball of A. Set $\tilde{A} \coloneqq A^\circ/A^{\circ\circ}$. Then $\tilde{A}$ is a discrete $\tilde{k}$-algebra.*

Lemma 1.5.4. *The set-theoretical map*

$$\begin{array}{rcl} Hom_{(k\text{-}Banach)}(k\{T_1, T_1^{-1}, \ldots, T_m, T_m^{-1}\}, k_S) & \to & (k_S^{\circ\times})^m \\ \phi & \mapsto & (\phi(T_1), \ldots, \phi(T_m)) \end{array}$$

induces the canonical bijective map

$$Hom(S, A_k^m(1,1)) \to (k_S^{\circ\times})^m,$$

where $A_k^m(1,1)$ is the torus $\mathscr{M}(k\{T_1, T_1^{-1}, \ldots, T_m, T_m^{-1}\})$.

In the case $m = 1$, it is the unit circle $\mathrm{A}_k^1(1,1) = \mathscr{M}(k\{T_1, T_1^{-1})$. The symbol A indicates the term “annulus”. In fact we want to use the symbol T for the term “torus’ and the symbol S^1 for the unit circle in the case $m = 1$, but the symbols are confusing. The symbols S and T are reserved for polytopes.

Proof. We have only to show that the map

$$\begin{array}{rcl} \mathrm{Hom}(S, \mathrm{A}_k^1(1,1)) & \to & k_S^{\circ\times} \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is bijective, because

$$k\{T_1, T_1^{-1}, \ldots, T_m, T_m^{-1}\} \cong_k k\{T_1, T_1^{-1}\}\hat{\otimes}_k \cdots \hat{\otimes}_k k\{T_m, T_m^{-1}\}.$$

It follows from the general fact of a bounded k-homomorphism of k-Banach algebras. For a uniform k-Banach algebra A, it is well-known that the map

$$\begin{array}{rcl} \mathrm{Hom}_{(k\text{-Banach})}(k\{T_1, T_1^{-1}\}, A) & \to & A \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is injective and its image coincides with the subset $\{f \in A^\times \mid \|f\| = \|f^{-1}\| = 1\} \subset A$. □

Lemma 1.5.5. *The set-theoretical map*

$$\begin{array}{rcl} Hom_{(k\text{-}Banach)}(k\{d^{(1)-1}T, d^{(-1)}T^{-1}\}, k_S) & \to & \left\{ f \in k_S^{\times m} \,\middle|\, \|f_i^\sigma\|_S \leq d_i^{(\sigma)}, {}^\forall i = 1, \ldots, m, {}^\forall \sigma = \pm 1 \right\} \\ \phi & \mapsto & (\phi(T_1), \ldots, \phi(T_m)) \end{array}$$

induce the canonical bijective maps

$$Hom(S, A_k^m(d^{(-1)}, d^{(1)})) \to \left\{ f \in k_S^{\times m} \,\middle|\, \|f_i^\sigma\|_S \leq d_i^{(\sigma)}, {}^\forall i = 1, \ldots, m, {}^\forall \sigma = \pm 1 \right\}$$

and

$$Hom(S, \mathring{A}_k(d^{(-1)}, d^{(1)})) \to \left\{ f \in k_S^{\times m} \,\middle|\, \|f_i^\sigma\|_R < d_i^{(\sigma)}, {}^\forall i = 1, \ldots, m, {}^\forall \sigma = \pm 1 \right\},$$

where $A_k^m(d^{(-1)}, d^{(1)})$ and $\mathring{A}_k^m(d^{(-1)}, d^{(1)})$ are the annuli $\mathscr{M}(k\{d_i^{(1)-1}T_i, d_i^{(-1)}T_i^{-1} \mid i = 1, \ldots, m\})$ and $\varinjlim \mathscr{M}(k\{d_i'^{(1)-1}T_i, d_i'^{(-1)}T_i^{-1} \mid i = 1, \ldots, m\})$ respectively, and where $d_i'^{(-1)}$ and $d_i'^{(1)-1}$ in the limit run through all pair $(d_i'^{(-1)}, d_i'^{(1)-1}) \in (0, \infty)^2$ such that $d_i^{(-1)} < d_i'^{(-1)} \leq d_i'^{(1)-1} < d_i^{(1)-1}$.

Corollary 1.5.6. *The bijective maps above induce the canonical bijective map*

$$Hom(S, \mathbb{G}_{m,k}) \cong k_S^\times,$$

where $\mathbb{G}_{m,k}$ is the open torus $\cup\mathscr{M}(k\{d^{-1}T_1, d'T_1^{-1}\})$.

Proof. Trivial because

$$\mathbb{G}_{m,k} = \varinjlim_{0<d'<d} \mathscr{M}(k\{d^{-1}T_1, d'T_1^{-1}\}).$$

□

We want to determine the multiplicative group $k_S^\times$ and $k_S^{\circ\times}$ for a k-affinoid simplex S_k. We prepare some basic facts. In order to determine the unit group of a k-Banach algebra A with a multiplicative norm, we often use the reduction functor $A \rightsquigarrow \tilde{A}$, but what we want to analyse is a uniform ring k_S, whose Gauss norm is not a multiplicative one. Moreover, although the reduction functor is sometimes used if we only see $A^{\circ\times}$, the reduced algebra $\tilde{A}$ is not necessarily so good one. We only deal with a thick polytope now. If one wants to deal with an affinoid simplex associated with a polytope which is not thick, it is suffices to fix an isomorphic integral affine map.

Lemma 1.5.7. *Let $S \subset \mathbb{R}^n$ be a thick polytope containing $(0,\ldots,0) \in \mathbb{R}^n$. Then the residue ring $\widetilde{k_S}$ is isomorphic to a $\tilde{k}$-algebra of the form*

$$A := \tilde{k}[T_x \mid x \in E_{k,n}]/(T_1 - 1, T_x T_y - a_{x,y} T_{xy} \mid x, y \in E_{k,n}),$$

where $a \colon E_{k,n} \times E_{k,n} \to \tilde{k}$ is a map satisfying $a_{x,y} = a_{y,x}$ and $a_{x,y} a_{xy,z} = a_{x,yz} a_{y,z}$ for any $x, y, z \in E_{k,n}$, and $a_{x,y} = 0$ unless $\|x\|_S, \|y\|_S \in |k^\times|$ and $\|xy\|_S = \|x\|_S \|y\|_S$.

Proof. Since S contains $(0,\ldots,0)$, we know $\|x\|_S \geq |x(0,\ldots,0)| = 1$ for any $x \in E_{k,n}$. Fix a system $c \colon E_{k,n} \to k \colon x \mapsto c_x$ such that $|c_x| = \|x\|_S \in [1,\infty)$ if $\|x\|_S \in |k^\times|$ and $c_x = 0$ if $\|x\|_S \notin |k^\times|$. Set $a_{x,y} := \overline{c_{xy}^{-1} c_x c_y} \in \tilde{k}$ if $\|x\|_S, \|y\|_S \in |k^\times|$ and $\|xy\|_S = \|x\|_S \|y\|_S$, and $a_{x,y} = 0$ if else. The map a satisfies the given condition. Define the ring homomorphism $\phi \colon k_S^\circ \to A$ by

$$\phi\left(\sum_{x \in E_{k,n}} f_x x\right) = \sum_{x \in E_{k,n}} \overline{f_x c_x} T_x.$$

The kernel of ϕ coincides with $k_S^{\circ\circ}$. Obviously ϕ is surjective, and hence we have done. □

The reduction $\widetilde{k_S}$ is not so simplified. Instead of it, we introduce the new idea of the reduction.

Definition 1.5.8. *Let $S \subset \mathbb{R}^n$ be a thick polytope. Then the strict residue ring $\breve{k_S}$ of k_S is a $\tilde{k}$-algebra*

$$\breve{k_S} := (k_S)^\circ / ((k_S)^{\circ\circ} + (k^{\circ\circ})_S),$$

where $(k^{\circ\circ})_S$ is an ideal $\{f \in (k_S)^\circ \mid f_x \in k^{\circ\circ}, x \in E_{k,n}\} \subset (k_S)^\circ$.

Definition 1.5.9. *Set $E_1 := \{x \in E_{k,1} \mid |x(1)| = 1\}$. Note that for a thick polytope $S \subset \mathbb{R}^n$ containing $(0,\ldots,0) \in \mathbb{R}^n$, one has*

$$\begin{aligned} (E_1)^n &= \left\{ x \in E_{k,n} \,\middle|\, |x(t)| = 1, {}^\forall t \in S \right\} \\ &= \left\{ x \in E_{k,n} \,\middle|\, \|x\|_S = \|x^{-1}\|_S = 1 \right\}. \end{aligned}$$

Lemma 1.5.10. *Let $S \subset \mathbb{R}^n$ be a thick polytope containing $(0,\ldots,0) \in \mathbb{R}^n$. Then the strict residue ring $\breve{k_S}$ is isomorphic to a $\tilde{k}$-algebra of the form*

$$A := \tilde{k}[T_x \mid x \in (E_1)^n]/(T_1 - 1, T_x T_y - T_{xy} \mid x, y \in (E_1)^n)$$

by the canonical homomorphism

$$(k_S)^\circ \to A \colon f \to \sum_{x \in E_{k,n}} \overline{f_x} T_x.$$

The homomorphism makes sense because the right hand side is an essetially finite sum of elements of the form $\overline{f_x} T_x$ for an element $x \in (E_1)^n$. Remark that if $\|x\|_S > 1$, then $\overline{f_x} = 0 \in \tilde{k}$ because $\|f\|_S = 1$.

Proof. The homomorphism is surjective and the kernel of the given homomorphism is obviously $(k_S)^{\circ\circ} + (k^{\circ\circ})_S$. □

Using the idea of the strict reduction $k_S \rightsquigarrow \breve{k}_S$, the unit group of k_S is easily calculated as below:

Proposition 1.5.11. *Let $S \subset \mathbb{R}^n$ be a thick polytope containing $(0,\ldots,0) \in \mathbb{R}^n$. An element $f \in (k_S)^\circ$ is invertible in $(k_S)^\circ$ if and only if there uniquely exists an element $x \in E_{k,n}$ such that $|f_x| = 1, \|x\|_S = \|x^{-1}\|_S = 1$, and $\|f_x^{-1}x^{-1}f - 1\|_S < 1$. In other words, the canonical homorphism*

$$\begin{array}{rcl} k^{\circ\times} \times (E_1)^n \times (1 + (k_S)^{\circ\circ}) & \to & (k_S)^{\circ\times} \\ (a, x, 1+g) & \mapsto & ax(1+g) \end{array}$$

is an isomorphism. Moreover if the valuation of k is discrete and the residue field $\tilde{k}$ is perfect, one has

$$\begin{array}{rcl} [\tilde{k}^\times] \times (E_1)^n \times (1 + (k_S)^{\circ\circ}) & \cong & (k_S)^{\circ\times} \\ (a, x, 1+g) & \mapsto & ax(1+g), \end{array}$$

where $[\cdot] \colon \tilde{k}^\times \to k^{\circ\times}$ is the Teichmüller embedding.

Proof. The sufficiency is trivial because k_S is complete. Take an arbitrary $f \in (k_S)^{\circ\times}$. Since f is invertible, its image $\overline{f}$ by the strict reduction map $(k_S)^\circ \to \breve{k}_S$ is invertible in $\breve{k}_S \cong_{\tilde{k}} A$. There exists some $x \in E_{k,n}$ such that $|f_x| = 1$, $\|x\|_S = \|x^{-1}\|_S = 1$, and either $|f_y|\|y\|_S < 1$ or $|f_y| < 1$ for any $y \neq x \in E_{k,n}$, because $A^\times = \{aT_x \in A \mid a \in \tilde{k}^\times, x, x^{-1} \in M_S\}$. Set $F := \{y \in E_{k,n} \mid |f_y|\|y\|_S = 1\}$. Since $\lim_{y \in E_{k,n}} |f_y|\|y\|_S = 0$, we know F is a finite set containing x. Let $M \subset E_{k,n}$ be the $\mathbb{Z}$-submodule generated by F, and fix a total order of M compatible with its group structure as we have done in the proof of Proposition 1.1.39. We want to prove the cardinarity of F is 1. Assume F has two or more elements. A total order is a total order also with respect to its reverse order, and therefore we may and do assume x is not the greatest element of F without loss of generality replacing its total order to its reverse order. Let $x' > x$ be the greatest element of F. Denote by g the inverse element $f^{-1} \in (k_S)^{\circ\times}$, and set $G := \{y \in E_{k,n} \mid |g_y|\|y\|_S = 1\}$. The finite set $FG := \{yz \mid y \in F, z \in G\}$ contains 1 because $x^{-1} \in G$. Since $x' > x$, one has $x'x^{-1} > 1 \in FG$. Since $(fg)_{x'x^{-1}} = 1_{x'x^{-1}} = 0$, there exists another element $x_1 \in F$ than x' such that $x'x^{-1}x_1^{-1} \in G$. Then one has $x'^2x^{-1}x_1^{-1} > x'x^{-1} > 1 \in FG$. Repeating this process, there exist some $x_2, x_3, \ldots < x' \in F$ such that $x'^2x^{-1}x_1^{-1}x_2^{-1}, x'^3x^{-1}x_1^{-1}x_3^{-1}, \ldots \in G$. Then one has a non-stable increasing sequence $1 < x'x^{-1} < x'^2x^{-1}x_1^{-1} < x'^3x^{-1}x_1^{-1}x_2^{-1} < \cdots \in FG$, and it contradicts the fact FG is a finite set. It follows F has no more than one element x. We have done. □

Corollary 1.5.12. *The set-theoretical bijective map*

$$Hom_{(k\text{-}Banach)}(k\{T_1, T_1^{-1}\}, k_S) \quad \to \quad (k_S)^{\circ\times}$$

$$\phi \mapsto \phi(T_1)$$

and the canonical isomorphism

$$\begin{array}{rcl} k^{\circ\times} \times (E_1)^n \times (1 + (k_S)^{\circ\circ}) & \to & (k_S)^{\circ\times} \\ (a, x, 1+g) & \mapsto & ax(1+g) \end{array}$$

induce the canonical bijective map

$$Hom(S, A^1_k(1,1)) \to k^{\circ\times} \times (E_1)^n \times (1 + (k_S)^{\circ\circ}).$$

Moreover if the valuation of k is discrete and the residue field $\tilde{k}$ is perfect, one has

$$[\tilde{k}^\times] \times (E_1)^n \times (1 + (k_S)^{\circ\circ}) \cong Hom(S, A^1_k(1,1)).$$

In particular the unit group of $(k_{[0,1]^n})^\circ$ has bee calculated. In order to determine the unit group $(k_{[0,1]^n})^\times$, one has to repeat a similar but much more complicated process.

Proposition 1.5.13. *Suppose* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} < \infty$. *Let $S \subset \mathbb{R}^n$ be a thick polytope containing* $(0, \ldots, 0) \in \mathbb{R}^n$. *An element $f \in k_S$ is invertible if and only if there uniquely exists an $x \in E_{k,n}$ such that $|f_x| \|x\|_S = \|f\|_S$ and $\|f_x^{-1} x^{-1} f - 1\|_S < 1$. In other words, the canonical homorphism*

$$\begin{array}{rcl} k^\times \times E_{k,n} \times (1 + (k_S)^{\circ\circ}) & \to & (k_S)^\times \\ (a, x, 1+g) & \mapsto & ax(1+g) \end{array}$$

is an isomorphism. Moreover if the valuation of k is discrete and the residue field $\tilde{k}$ is perfect, one has

$$\begin{array}{rcl} [\tilde{k}^\times] \times \pi_k^{\mathbb{Z}} \times E_{k,n} \times (1 + (k_S)^{\circ\circ}) & \cong & (k_S)^\times \\ (a, \pi_k^m, x, 1+g) & \mapsto & a\pi_k^m x(1+g), \end{array}$$

Proof. The sufficiency is trivial because k_S is complete. Take an arbitrary $f \in (k_S)^\times$. To begin with, we prove there uniquely exists an $x \in E_{k,n}$ such that $|f_x| = \max_{y \in E_{k,n}} |f_y|$ and $|f_x| > |f_y|$ for any $y \neq x \in E_{k,n}$. Since S contains $(0, \ldots, 0)$, we know $\|x\|_S \geq 1$ for any $x \in E_{k,n}$. By the definition of k_S, one has $\lim_{x \in E_{k,n}} |f_x| \leq \lim_{x \in E_{k,n}} |f_x| \|x\|_S = 0$. Therefore the subset $\{|f_x| \mid x \in E_{k,n}\} \subset [0, \infty)\}$ has a non-zero greatest element because $f \neq 0$. Set $F := \{x \in E_{k,n} \mid |f_x| = \max_{y \in E_{k,n}} |f_y|\}$, and then F is a finite set. Similarly, set $g := f^{-1}$ and $G := \{x \in E_{k,n} \mid |g_x| \geq (\max_{y \in E_{k,n}} |f_y|)^{-1}\}$. Since $fg = 1$, we know $1 \in FG := \{xy \mid x \in F, y \in G\}$. Indeed, if $1 \neq FG$, then $|f_x||g_{x^{-1}}| < (\max_{y \in E_{k,n}} |f_y|)(\max_{y \in E_{k,n}} |f_y|)^{-1} = 1$ for any $x \in E_{k,n}$ and hence $1 = |1| = |\sum_{x \in E_{k,n}} f_x g_{x^{-1}}| < 1$ because $\lim_{x \in E_{k,n}} f_x g_{x^{-1}} = 0$. It is a contradiction. Take an $x \in F$ satisfying $x^{-1} \in G$. Fix a total order of the finitely generated (free) $\mathbb{Z}$-submodule $M \subset E_{k,n}$ generated by F and G compatible with its group structure as we did in the proof of Proposition 1.1.39. Let $x' \geq x \in F$ and $x'' \geq x^{-1} \in G$ be the greatest elements. For any $(y, z) \neq (x', x'') \in E_{k,n} \times E_{k,n}$ satisfying $yz = x'x''$, one

has $y \notin F$ or $z \notin G$ by the choice of x' and x'', and hence $|f_y||g_z| < |f_{x'}||g_{x''}|$. It implies $|1_{x'x''}| = |(fg)_{x'x''}| = |f_{x'}||g_{x''}| > 0$, i.e. $x'x'' = 1 = xx^{-1}$. Therefore $x' = x$ and $x'' = x^{-1}$ by the choice of x' and x''. On the other hand, take the least element $x''' \leq x \in F$ and $x'''' \leq x^{-1} \in G$. By the totally same argument, one obtains $x''' = x$ and $x'''' = x^{-1}$. Hence $F = \{x\}$ and $G = \{x^{-1}\}$. In addition by the argument of the absolute value of coefficients above, one has obtained the equality $|f_x||g_{x^{-1}}| = 1$.

Now we prove $|f_x^{-1}x^{-1}f - 1|_S < 1$, i.e. $|(x^{-1}f)_y|||y||_S = |f_{xy}|||y||_S < |f_x| = |(x^{-1}f)_1|$ for any $y \neq 1 \in E_{k,n}$. Embedding k_S into C_S and repelacing k to C, we may and do assume k is algebraically closed and $E_{k,n}$ is a $\mathbb{Q}$-vector space. Remark that the second implication of this proposition directly follows from the first implication, and hence one does not have to warry about the fact that C is not a discrete valuation field.

Assume $\max_{y \neq 1 \in E_{k,n}} |f_{xy}|||y||_S \geq |f_x|$, and set $U := \{y \in E_{k,n} \mid y \neq 1, |f_{xy}|||y||_S = \max_{y' \in E_{k,n}} |f_{xy'}|||y'||_S\} \neq \emptyset$. Then U is a finite set because $|f_{xz}|||z||_S = |(x^{-1}f)_z|||z||_S \to 0$. Since $|f_xy| < |f_x|$, one has $||y||_S > 1$ for any $y \in U$. Take a $y_0 \in U$ such that $|f_{xy_0}| = \max_{y \in U} |f_{xy}|$. Set $V := \{y \in E_{k,n} \mid ||y||_S = 1\}$, and fix a $\mathbb{Q}$-basis $\{y_\lambda \mid \lambda \in \Lambda\}$ of V. Take a complementary space $W \in E_{k,n}$ of V containing y_0. Since $\dim_{\mathbb{Q}} \sqrt{|k^\times|} < \infty$, W is of finite dimension. Take a $\mathbb{Q}$-basis $\{y_0, \ldots, y_k\}$ of W containing y_0. The system $\{y_\lambda \mid \lambda \in \Lambda' := \Lambda \sqcup \{0, \ldots, k\}\}$ forms a $\mathbb{Q}$-basis of $E_{k,n}$. Set $U' := \{z \in U \mid |f_{xz}| = |f_{xy_0}|, y_0^{-1}z \in V\} = \{y_0, z_1, \ldots, z_m\}$, and $z_i = y^{I_i} := \Pi y_\lambda^{I_{i\lambda}}$ by the unique multi-index $I_i = (I_{i\lambda})_{\lambda \in \Lambda'} \in \mathbb{Q}^{\oplus \Lambda'}$. ($I_{i0} = 1$ and $I_{i1} = \cdots = I_{ik} = 0$ by the definition of U'.) Then the set $L := \{\lambda \in \Lambda \mid I_{i\lambda} \neq 0, {}^\exists i = 1, \ldots, m\}$ is a finite set, and set $L = \{l_1, \ldots, l_j\}$ and $J_i = (J_{i1}, \ldots, J_{ij}) := (I_{il_1}, \ldots, I_{il_j}) \in \mathbb{Q}^j$ so that $z_i = y_0 y_l^{J_i} := y_0 y_{l_1}^{J_{i1}} \cdots y_{l_j}^{J_{ij}}$ for each $i = 1, \ldots, m$. Since z_i's are distinct, so are the multi-indices J_i's. Consider the polynomial

$$P(T_1, \ldots, T_j) := 1 + \sum_{i=1}^{m} \frac{f_{xz_i}}{f_{xy_0}} T^{J_i} = 1 + \sum_{i=1}^{m} \frac{f_{xz_i}}{f_{xy_0}} T_1^{J_{i1}} \cdots T_j^{J_{ij}}.$$

Since $|f_{xy_0}^{-1} f_{xz_1}| = \cdots = |f_{xy_0}^{-1} f_{xz_i}| = 1$, the residue $\overline{P} \in \tilde{k}[T_1, \ldots, T_j]$ is the sum of m distinct monomials. Since $\tilde{k}$ is algebraically closed, it is an infinite field. Hence there exists some $\overline{u} = (\overline{u_1}, \ldots, \overline{u_m}) \in \tilde{k}$ such that $\overline{P}(\overline{u}) \neq 0 \in \tilde{k}$, and take a lift $u = (u_1, \ldots, u_m) \in k^{\circ\times}$ of $\overline{u}$. By the choice of $\overline{u}$, one has $|P(u)| = 1$. Now the image of k-algebra homomorphism

$$\begin{aligned} \phi \colon k_S &\to k[[T_i^q \mid i = 0, \ldots, m, q \in \mathbb{Q}]] \\ h &\mapsto \sum_{I \in \mathbb{Q}^{m+1}} \left(\sum_{J \in \mathbb{Q}^{\oplus \Lambda}} h_{y_0^{I_0} \cdots y_k^{I_k} y^J} u^{J_l} \right) T^q \end{aligned}$$

is contained in the k-subalgebra

$$k\{W\} := \{F(T1, \ldots, T_k) \in k[[T_i^q \mid i = 0, \ldots, m, q \in \mathbb{Q}]] \mid \lim_{|I| \to \infty} |F_I|||y_0^{I_0} \cdots y_k^{I_k}||_S = 0\},$$

where y^I and u^{J_l} are the essentially finite products $\Pi_{\lambda \in \Lambda} y_\lambda^{I_\lambda}$ and $\Pi_{i=1}^m u_i^{J_{\lambda_{l_i}}}$. Equiping $k\{W\}$ a power-multiplicative norm

$$|| \cdot ||_S \colon k\{W\} \to [0, \infty)$$

$$F \mapsto \max_{I \in \mathbb{Q}^{k+1}} |F_I| \| y_0^{I_0} \cdots y_k^{I_k} \|_S,$$

we regard $k\{W\}$ as a k-Banach algebra and ϕ as a bounded k-homomorphism restricting its codomain. Set $F := \phi(x^{-1} f) \in k\{y_0^{\mathbb{Q}}\}^{\times}$, and then $\|F\|_S = |F_{(1,0,\ldots,0)}| \|y_0\|_S = |f_{xy_0}| \|y_0\|_S \geq |f_x| = |F_{(0,\ldots,0)}| \|1\|_S$. Indeed, for any $z \in y_0 V \backslash U'$ one has either $|f_{xz}| \|y_0\|_S = |f_{xz}| \|z\|_S < |f_{xy_0}| \|y_0\|_S$ or $|f_{xz}| < |f_{xy_0}|$ by the definition of U and U'. It follows $|f_{xy_0 y^J} u^{J_l}| < |f_{xy_0}|$ for any $J \in \mathbb{Q}^{\oplus \Lambda}$ satisfying $y_0 y^J \notin U'$. On the other hand, by the definition of u, one has

$$\left| \sum_{\substack{J \in \mathbb{Q}^{\oplus \Lambda} \\ y_0 y^J \in U'}} f_{xy_0 y^J} u^{J_l} \right| = |f_{xy_0} P(u)| = |f_{xy_0}| |P(u)| = |f_{xy_0}|,$$

and hence

$$|F_{(1,0,\ldots,0)}| = \left| \sum_{J \in \mathbb{Q}^{\oplus \Lambda}} f_{xy_0 y^J} u^{J_l} \right| \|y_0\|_S = |f_{xy_0}| \|y_0\|_S.$$

Furthermore, one has

$$|F_{(0,\ldots,0)}| \|1\|_S = \left| \sum_{J \in \mathbb{Q}^{\oplus \Lambda}} f_{xy^J} u^{J_l} \right| \|1\|_S = \left| f_x + \sum_{J \neq 0 \in \mathbb{Q}^{\oplus \Lambda}} f_{xy^J} u^{J_l} \right| = |f_x|$$

and

$$\begin{aligned} & \left| \sum_{J \in \mathbb{Q}^{\oplus \Lambda}} f_{xy_0^{I_0} \cdots y_k^{I_k} y^J} u^{J_l} \right| \| y_0^{I_0} \cdots y_k^{I_k} \|_S \leq \max_{J \in \mathbb{Q}^{\oplus \Lambda}} |f_{xy_0^{I_0} \cdots y_k^{I_k} y^J}| \| y_0^{I_0} \cdots y_k^{I_k} \|_S \\ = \; & \max_{J \in \mathbb{Q}^{\oplus \Lambda}} |f_{xy_0^{I_0} \cdots y_k^{I_k} y^J}| \| y_0^{I_0} \cdots y_k^{I_k} y^J \| \leq \|x^{-1} f\|_S = |f_{xy_0}| \|y_0\| \end{aligned}$$

for any $I \in \mathbb{Q}^{k+1}$. We conclude $\|F\|_S = |F_{(1,0,\ldots,0)}| \|y_0\|_S = |f_{xy_0}| \|y_0\|_S \geq |f_x| = |F_{(0,\ldots,0)}| \|1\|_S$. Now replacing S to its convex closure, we assume S is convex and arcwise connected. Since $|F_{(1,0,\ldots,0)}| \|y_0\|_S = |f_{xy_0}| \|y_0\|_S \geq |f_x| = |F_{(0,\ldots,0)}| \|1\|_S > |f_{xy_0}| = |f_{xy_0}| |xy_0(0, \ldots, 0)|$, there exists some point $t \in S$ such that $\|F\|_t = |F_{(1,0,\ldots,0)}| |y_0(t)| = |F_I| |y_0^{I_0} \cdots y_k^{I_k}(t)|$ for some $I \neq (1, 0, \ldots, 0) \in \mathbb{Q}^{k+1}$, where $\|F\|_t$ is given by the bounded multiplicative norm

$$\begin{aligned} \| \cdot \|_t \colon k\{W\} & \to [0, \infty) \\ F' & \mapsto \max_{I \in \mathbb{Q}^{k+1}} |F'_I| |y_0^{I_0} \cdots y_k^{I_k}(t)|. \end{aligned}$$

Denote by $k\{W\}_t$ the completion of $k\{W\}$ with respect to the valuation $\| \cdot \|_t$. Then $F \in k\{W\}_t^{\times}$ because $F \in k\{W\}^{\times}$. Fix a sufficiently large algebraically closed extension $\mathbb{C}/k$ of complete non-Archimedean fields equiped with a surjective valuation $| \cdot | \colon \mathbb{C} \twoheadrightarrow [0, \infty)$, and take a group-theoretical section $\mathbb{R}_+^{\times} \to \mathbb{C}^{\times} \colon s \mapsto p^{\log_{|p|}(s)}$ of the valuation. Then

$p^{-log_{|p|}\|F\|_t}F \in Q(\mathbb{C}\{W\}_t)^\circ$ is invertible in $Q(\mathbb{C}\{W\}_t)^\circ$ because $\|\cdot\|_t$ is a valuation, where $Q(\mathbb{C}\{W\}_t)$ is the field of fraction of $\mathbb{C}\{W\}_t$, and therefore the image of $p^{-log_{|p|}\|F\|_t}F$ in the residue field $Q(\widetilde{\mathbb{C}\{W\}_t})$ is also invertible. The $\tilde{\mathbb{C}}$-homomorphism

$$Q(\widetilde{\mathbb{C}\{W\}_t}) \to \tilde{\mathbb{C}}[T_i^q \mid i = 0, \ldots, k, q \in \mathbb{Q}]$$

induced by the surjective homomorphism

$$\begin{array}{rcl} \mathbb{C}\{W\}_t^\circ & \to & \tilde{\mathbb{C}}[T_i^q \mid i = 0, \ldots, k, q \in \mathbb{Q}] = \varinjlim_{r \in \mathbb{N}_+} \tilde{\mathbb{C}}[T_0^{\frac{1}{r}}, T_0^{-\frac{1}{r}}, \ldots, T_k^{\frac{1}{r}}, T_k^{-\frac{1}{r}}] \\ F' & \mapsto & \max_{I \in \mathbb{Q}^{k+1}} \overline{F'_I p^{\log_{|p|} |y_0^{I_0} \cdots y_k^{I_k}(t)|}} T_0^{I_0} \cdots T_k^{I_k} \end{array}$$

is an isomorphism. However, the image of $p^{-log_{|p|}\|F\|_t}F$ in the residue field $\tilde{\mathbb{C}}[T_i^q \mid i = 0, \ldots, k, q \in \mathbb{Q}]$ is not a monomial because of the condition $\|F\|_t = |F_{(1,0,\ldots,0)}||y_0(t)| = |F_I||y_0^{I_0} \cdots y_k^{I_k}(t)|$, and hence not invertible. It's a contradiction. Therefore we conclude $\max_{y \neq 1 \in E_{k,n}} |f_{xy}|\|y\|_S < |f_x|$, and we have done. $\square$

Corollary 1.5.14. *Suppose* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} < \infty$. *The bijective maps above induce the canonical bijective map*

$$Hom(S, \mathbb{G}_{m,k}) \cong k_S^\times = k^\times \times E_{k,n} \times (1 + (k_S)^{\circ\circ}).$$

Moreover if the valuation of k is discrete and k is perfect, then one has

$$[\tilde{k}^\times] \times \pi_k^{\mathbb{Z}} \times E_{k,n} \times (1 + (k_S)^{\circ\circ}) \cong Hom(S, \mathbb{G}_{m,k}),$$

where $\pi_k \in k$ *is a uniformizer of* k.

We finish the examples.

2 Overconvergence

In this section, we deal with overconvergent analytic functions. There seems to be two ways of considering an overconvergent structure of a rigid analytic space: arranging the information of the underlying topological space of an analytic space, or arranging the infomation of the structure sheaf. The first one is the idea of the use of a k-germ defined in [BER2]. A k-germ has an information of a neighbourhood of its underlying space, and therefore it is easily equiped with the overconvergent structure in a natural way: just consider analytic functions which can be extended to a neighbourhood of the domain. There is a fully faithful functor from the category of k-analytic spaces to the category of k-germs. The image of a closed analytic space (i.e. an analytic space with no boundary, [BER1]) by the functor seems to have the required overconvergent structure, but images of affinoid spaces are different from what we want even if they are rig-smooth. For example, the image of the closed unit disc D_k^1 is the k-germ $[\mathrm{D}_k^1, \mathrm{D}_k^1]$, but what seems to be an

intuitively collect object is the k-germ $[\mathbb{A}^1_k, \mathrm{D}^1_k]$. We failed to construct good overconvergent struucture of k-germ resolving this "non-closed case problem". The other one is the idea of the use of a k-daggaer space, which is introduced by Elmar Grosse-Kloènne in [KLO1]. A k-dagger space is a G-topological space whose underlying topological space is that of a rigid k-analytic space in the sense of [BGR] and which is equiped with a dense subsheaf of a finitely generated weakly complete k-algebra of the structure sheaf as a k-analytic space on what is called the strong G-topology of it. For the terminology of a k-dagger space and a general weakly complete k-algebra, see [KLO1]. We introduce more general definition of k-dagger space corresponding to a (not necessarily strict) k-analytic space, and distinguish it from the previous one, which we will call a strict k-dagger space. There is a fully faithful functor from the category of strict k-dagger spaces to the cateogry of rigid analytic spaces. The category of partially proper strict k-dagger spaces is equivalent with the category of partially proper rigid analytic spaces by [KLO1], 2.27, and the category of paracompact Hausdorff strictly k-analytic space is equivalent with the category of quasi-separated rigid k-analytic spaces which admit an admissible affinoid covering of finite type by [BER2], 1.6.1. Note that rigid Stein spaces and closed rigid analytic spaces are partilly proper, and therefore we can regard a Hausdorff strict k-analytic space which is a Stein space or a paracompact smooth space as a k-dagger space. For more detail, see [KLO1], [KLO2], and [KLO3].

2.1 Weak completeness

In this subsection, we introduce the notion of weak completeness of a normed k-algebra. The original notion of weak completeness is refered to for an abstract algebra called "an (R, I)-algebra" by P. Monsky and G. Wachnitzer in [MW], and such an algebra is canonically endowed with the structure of a topological ring regarded as an adic ring in the sense of [HUB]. Roland Huber used an adic ring in a certain class, called an affinoid ring in his works. Of course the term "affinoid ring" indicates the object different from the affinoid algebra we are dealing with in this paper, but is heavily related with it. For the terminology of an adic ring, see [HUB]. Now, we want to make use of the notion of weak completeness for a normed k-algebra. Though an adic ring over (k, k°) does not necessarily admit a structure of a normed k-algebra, an affinoid ring (A, A^+) satisfying the following handy conditions is naturally regarded as a uniformly normed k-algebra. The conditions are that A is a Hausdorff topological k-algebra, that $A^{\circ} \cap k = k^{\circ}$, that $A^{\circ} \subset A$ is open, that $A^+ = A^{\circ}$, and that $A^{\circ\circ} \subset A^{\circ}$ is an ideal of the definition of A. Therefore we extend the notion of weak completeness to the class of uniformly normed k-algebras. The condition for the weak completeness makes sense if one rejects the assumption of the uniformity, and hence we state the definition of the weak completeness of a normed k-algebra. To begin with, before extending the notion of the weak completeness to the class of normed k-algebras, we introduce the most basic example of an weakly complete normed algebra called a Monsky-Washnitzer algebra, which is deeply studied in the various works by Elmar Grosse-Klönne. We will not prove lemmas for the well-known facts and the straightforward facts about weak completeness and dagger algebras. Almost all

properties of weak completeness of a normed k-algebra is verified in the totally same way as that of an (R, I)-algebra.

Definition 2.1.1. *For an integer $m \in \mathbb{N}$ and a parametre $r = (r_1, \ldots, r_m) \in (0, \infty)^m$, set*

$$k\{r^{-1}T\}^\dagger = k\{r_1^{-1}T_1, \ldots, r_m^{-1}T_m\}^\dagger := \varinjlim_{r' \to r+} k\{r'^{-1}T\},$$

where the limit is the direct limit of the underlying k-algebras. By the exactness of the direct limit, the canonical k-algebra homomorphism $k\{r^{-1}T\}^\dagger \to k\{r^{-1}T\}$ is injective, and regard $k\{r^{-1}T\}^\dagger$ as a k-subalgebra of $k\{r^{-1}T\}$. Endow $k\{r^{-1}T\}^\dagger \subset k\{r^{-1}T\}$ the restriction of the Gauss norm of $k\{r^{-1}T\}$. Call $k\{r^{-1}T\}^\dagger$ a general Monsky-Washnitzer algebra. If $r = (1, \ldots, 1)$, then write $k\{T\}^\dagger = k\{T_1, \ldots, T_m\}^\dagger$ instead of $k\{r^{-1}T\}^\dagger$ for short, and call $k\{T\}^\dagger$ a Monsky-Washnitzer algebra.

This notation $k\{r^{-1}T\}^\dagger$ does not imply the existence a "dagger" operator $A \rightsquigarrow A^\dagger$ from an affinoid algebra to an weakly complete algebra. For example, if an affinoid algebra A has two distinct presentation $k\{r^{-1}T\}/I$ and $k\{r'^{-1}T'\}/I'$, the two associated k-algebra $k\{r^{-1}T\}^\dagger/(I \cap k\{r^{-1}T\}^\dagger)$ and $k\{r'^{-1}T'\}^\dagger/(I' \cap k\{r'^{-1}T'\}^\dagger)$ possibly differ from each other. Now such an algebra is endowed with two topologies. One is the relative topology of $k\{r^{-1}T\}$, which is the topology given by the Gauss norm. The other one is the underlying topology of the direct limit as a topological k-algebra. Be careful about the treatment of the direct limit. One has to distinguish the direct limits in the ten categories of k-algebras, of topological spaces, of topological k-vector spaces, of topological k-algebras, of seminormed k-algebras, of normed k-algebras, of k-Banach algebras, of uniformly seminormed k-algebras, of uniformly normed k-algebras, and of uniform k-Banach algebras from each other.

The categories of seminormed k-algebras, normed k-algebras, and k-Banach algebras do not admit the direct limit along an arbitrary direct system, while the other categories do. Consider the case the direct limit exists for a direct system. The underlying k-algebras of the direct limits in the categories of topological k-algebras and of uniformly seminormed k-algebras coincide with the direct limit of the underlying k-algebras. The underlying k-vector space of the direct limit in the category of k-algebras coincides with the direct limit of the underlying topological k-vector spaces, and the underlying set of the direct limit of the underlying k-algebras coincides with the underlying set of the direct limit of the underlying topological space. However, the underlying k-algebra of the direct limit in the other categories do not coincide with the direct limit of the underlying k-algebras in general. The direct limit of uniform ones in the category of not necessarily uniform ones is uniform and hence coincides with the direct limit of them in the category of uniform ones. Each of the underlying topology of the direct limit in the category of topological k-vector spaces, the underlying topology of the direct limit in the category of topological k-algebras, and the direct limit topology, which is the topology of the direct limit in the category of topological spaces, is much finer than the underlying topology of the direct limit in the category of uniformly seminormed k-algebras. One marely uses the direct limit topology because it does not guarantee the continuity of the addition and

the multiplication. Similarly the topology as the direct limit in the category of topological k-vector spaces is not interesting. The topology as the direct limit in the category of topological k-algebra works well, but it is hard to handle and analyse. Therefore the most suitable topology is given by the direct limit seminorm. Now there is a canonical surjective k-algebra homomorphism from the direct limit of the underlying k-algebras to the underlying k-algebra of the direct limit in the category of uniformly normed k-algebras, which is not injective in general. It is injective if and only if the direct limit seminorm, which is the seminorm of the direct limit in the category of uniformly seminormed k-algebras, is a norm. Then the direct limit seminorm coincides with the direct limit norm, which is the norm of the direct limit in the category of uniformly normed k-algebras.

The k-algebra $k\{r^{-1}T\}^{\dagger}$ is the direct limit of the underlying k-algebra of multiplicative k-Banach algebras, and it coincides with the underlying k-algebra of the direct limits in the categories of topological k-algebras, of seminormed k-algebras, of normed k-algebras, of uniformly seminormed k-algebras, and of uniformly normed k-algebras. It does not coincide with the underlying k-algebra of the direct limits in the categories of k-Banach algebras or of uniform k-Banach algebras in general. The direct limit seminorm and the direct limit norm coincide with the restriction of the Gauss norm of $k\{r^{-1}T\}$. As we mentioned, the topologies other than the one given by the direct limit seminorm are bothersome. Therefore we endow $k\{r^{-1}T\}^{\dagger}$ with the topology given by the Gauss norm.

Definition 2.1.2 (formal partial differential)**.** *For integers $n, i \in \mathbb{N}$ satisfying $1 \leq i \leq n$, define the partial differential $\partial/\partial T_i \colon k[[T]] \to k[[T]] = k[[T_1, \ldots, T_n]]$ by setting*

$$\frac{\partial}{\partial T_i} F = \frac{\partial F}{\partial T_i} := \sum_{I \in \mathbb{N}^n} F_{I+\delta_i}(I_i + 1) T^I \in k[[T]]$$

for each $F = \sum_{I \in \mathbb{N}} F_I T^I \in k[[T]]$, where $\delta_i \in \mathbb{N}^n$ is the vector whose all entries except the i-th entry are 0 and whose i-th entry is 1.

Lemma 2.1.3 (Leibniz rule)**.** *For integers $n, i \in \mathbb{N}$ satisfying $1 \leq i \leq n$ and formal power series $F, G \in k[[T]] = k[[T_1, \ldots, T_n]]$, one has*

$$\frac{\partial FG}{\partial T_i} = F \frac{\partial G}{\partial T_i} + \frac{\partial F}{\partial T_i} G$$

Proof. Note that $k[[T]]$ is an integral domain. One has

$$\begin{aligned}
\frac{\partial FG}{\partial T_i} T_i &= T_i \frac{\partial}{\partial T_i} \sum_{I \in \mathbb{N}^n} \left(\sum_{J \leq I \in \mathbb{N}^n} F_{I-J} G_J \right) T^I \\
&= \sum_{I \in \mathbb{N}^n} \left(\sum_{J \leq I \in \mathbb{N}^n} F_{I-J} G_J \right) I_i T^I \\
&= \sum_{I \in \mathbb{N}^n} \left(\sum_{J \in \mathbb{N}^n} F_I G_J \right) (I_i + J_i) T^{I+J}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{I\in\mathbb{N}^n} F_I I_i T^I\right)\left(\sum_{J\in\mathbb{N}^n} G_J T^J\right) + \left(\sum_{I\in\mathbb{N}^n} F_I T^I\right)\left(\sum_{J\in\mathbb{N}^n} G_J J_i T^J\right) \\
&= F\frac{\partial G}{\partial T_i}T_i + \frac{\partial F}{\partial T_i}GT_i,
\end{aligned}
$$

and it follows

$$\frac{\partial FG}{\partial T_i} = F\frac{\partial G}{\partial T_i} + \frac{\partial F}{\partial T_i}G.$$

□

Lemma 2.1.4. *For parametres $r = (r_1, \ldots, r_n) \in (0, \infty)^n$, the partial differential preserves the k-subalgebras $k\{r^{-1}T\}^\dagger \subset k\{r^{-1}T\} \subset k[[T]] = k[[T_1, \ldots, T_n]]$. The restrictions $\partial/\partial T_i \colon k\{r^{-1}T\}^\dagger \to k\{r^{-1}T\}^\dagger$ and $\partial/\partial T_i \colon k\{r^{-1}T\} \to k\{r^{-1}T\}$ are bounded k-linear homomorphism of operator norm* 1.

Proof. Take a formal power series $F = \sum_{I\in\mathbb{N}^n} F_I T^I \in k[[T]]$ and suppose $F \in k\{r^{-1}T\}^\dagger$ or $F \in k\{r^{-1}T\}$. For any $i = 1, \ldots, n$, one has

$$|F_{I+\delta_i}(I_i + 1)|r^I \leq |F_{I+\delta_i}|r^I$$

and hence $\partial F/\partial T_i \in k\{r^{-1}T\}^\dagger$ or $\partial F/\partial T_i \in k\{r^{-1}T\}$ respectively. The operator norm is obviously 1 by the estimation above. □

Lemma 2.1.5. *The k-subalgebra $k\{r^{-1}T\}^\dagger \subset k\{r^{-1}T\}$ is the subset of a power-series $F = \sum_{I\in\mathbb{N}^m} F_I T^I \in k\{r^{-1}T\}$ such that there exists some $\delta \in (0, 1)$ such that $|F_I|r^I < \delta^{|I|}$ for sufficiently large $I \in \mathbb{N}^m$.*

Lemma 2.1.6. *The subset $k\{r^{-1}T\}^\dagger \subset k\{r^{-1}T\}$ is the k-subalgebra generated by power-series $F \in k\{r^{-1}T\}$ satisfying that there exist parametres $\delta \in (0, 1)$ and $\lambda \in (0, \infty)$, and a sequence $(P_d)_{d\in\mathbb{N}} \in k[T]^{\mathbb{N}}$ of polynomials such that* $\deg P_d \leq \lambda(d + 1)$ *and $\|P_d\| < \delta^d$ with respect to the Gauss norm of $k\{\|f\|^{-1}T\}$ for each $d \in \mathbb{N}$ and $F = \sum_{d\in\mathbb{N}} P_d(f) \in k\{r^{-1}T\}$.*

Corollary 2.1.7. *The follwing are equivalent for a normed k-algebra A:*

(i) for any $m \in \mathbb{N}$ and for any $f = (f_1, \ldots, f_m) \in (A\backslash\{0\})^m$, the k-algebra homomorphism $k[T] \to A \colon T \mapsto f$ is uniquely extended to a bounded k-algebra homomorphism $k\{\|f\|^{-1}T\}^\dagger \to A$;

(ii) for any $m \in \mathbb{N}$, for any $f = (f_1, \ldots, f_m) \in (A\backslash\{0\})^m$, for any $\delta \in (0, 1)$, and for any powerseries $F = \sum_{I\in\mathbb{N}^m} F_I T^I \in k\{\|f\|^{-1}T\} = k\{\|f_1\|^{-1}T_1, \ldots, \|f_m\|T_m\}$ such that $|F_I|\|f\|^I < \delta^{|I|}$ for each $I \in \mathbb{N}^m$, the infinite sum $\sum_{I\in\mathbb{N}^m} F_I f^I = \sum_{I\in\mathbb{N}^m} F_I f_1^{I_1} \cdots f_m^{I_m}$ converges in A; and

(iii) for any $m \in \mathbb{N}$, for any $f = (f_1, \ldots, f_m) \in (A\backslash\{0\})^m$, for any $\delta \in (0, 1)$, for any $\lambda \in (0, \infty)$, and for any sequence $(P_d)_{d\in\mathbb{N}} \in k[T]^{\mathbb{N}} = k[T_1, \ldots, T_m]^{\mathbb{N}}$ of polynomials such that $\deg P_d \leq \lambda(d + 1)$ *and $\|P_d\| < \delta^d$ with respect to the Gauss norm of $k\{\|f\|^{-1}T\}$ for each $d \in \mathbb{N}$, the infinite sum $\sum_{d\in\mathbb{N}} P_d(f)$ converges in A.*

Moreover, if the valuation of k is non-trivial and if A is uniform, the conditions (i)-(iii) are equivalent with each of

(i)' *for any $m \in \mathbb{N}$ and for any $f = (f_1, \ldots, f_m) \in (A^\circ)^m$, the k-homomorphism $k[T] \to A \colon T \mapsto f$ is uniquely extended to $k\{T\}^\dagger \to A$;*

(ii)' *for any $m \in \mathbb{N}$, for any $f = (f_1, \ldots, f_m) \in (A^\circ)^m$, for any $\delta \in (0,1)$, and for any powerseries $F = \sum_{I \in \mathbb{N}^m} F_I T^I \in k\{T\} = k\{T_1, \ldots, T_m\}$ such that $|F_I| < \delta^{|I|}$ for each $I \in \mathbb{N}^m$, the infinite sum $\sum_{I \in \mathbb{N}^m} F_I f^I = \sum_{I \in \mathbb{N}^m} F_I f_1^{I_1} \cdots f_m^{I_m}$ converges in A; and*

$(iii)'$ *for any $m \in \mathbb{N}$, for any $f = (f_1, \ldots, f_m) \in (A^\circ)^m$, for any $\delta \in (0,1)$, for any $\lambda \in (0,\infty)$, and for any sequence $(P_d)_{d \in \mathbb{N}} \in k[T]^{\mathbb{N}} = k[T_1, \ldots, T_m]^{\mathbb{N}}$ of polynomials such that $\deg P_d \leq \lambda(d+1)$ and $\|P_d\| < \delta^d$ with respect to the Gauss norm of $k\{T\}$ for each $d \in \mathbb{N}$, the infinite sum $\sum_{d \in \mathbb{N}} P_d(f)$ converges in A.*

Note that the conditions (ii) and (iii) are requivalent for a seminormed k-algebra. Using these equivalent conditions, we define the notion of weak completeness of a normed k-algebra.

Definition 2.1.8. *A seminormed k-algebra A is said to be weakly complete if it satisfies one of the conditions (ii) and (iii) in the previous lemma. A weakly complete seminormed k-algebra is said to be an weakly complete k-algebra if it is a normed k-algebra. If in addition $\rho(A) \subset \sqrt{|k^\times|} \cup \{0\} \subset [0,\infty)$, we say A is an weakly complete strict k-algebra, where ρ is the spectral radius*

$$\begin{array}{rcl} \rho \colon A & \to & [0,\infty) \\ f & \mapsto & \rho(f) \coloneqq \lim_{n \in \mathbb{N}} \|f^n\|^{1/n}. \end{array}$$

By the previpus lemma, a normed k-algebra is weakly complete if and only if it satisfies one of the conditions (i)-(iii).

Definition 2.1.9. *An weakly complete k-algebra A is said to be a finitely generated weakly complete k-algebra if there exist an integer $m \in \mathbb{N}$ and elements $f = (f_1, \ldots, f_m) \in (A \backslash \{0\})^m$ such that for any $g \in A$, there exist and a powerseries $F = \sum_{I \in \mathbb{N}^m} F_I T^I \in k\{\|f\|^{-1}T\}^\dagger = k\{\|f_1\|^{-1}T_1, \ldots, \|f_m\|^{-1}T_m\}^\dagger$ such that $g = \sum_{I \in \mathbb{N}^m} F_I f^I = \sum_{I \in \mathbb{N}^m} F_I f^I = \sum_{I \in \mathbb{N}^m} F_I f_1^{I_1} \cdots f_m^{I_m} \in A$.*

A k-Banach algebra is weakly complete. By the criterion for strictness, [BER1], 2.1.6, an affinoid algebra is an weakly complete strict k-algebra if and only if it is a strict k-affinoid algebra.

Lemma 2.1.10. *A k-algebra of the form $k\{r^{-1}T\}^\dagger$ is a Noetherian ring, and any ideal $I \subset k\{r^{-1}T\}^\dagger$ is closed.*

Definition 2.1.11. *For a proper ideal $I \subsetneq k\{r^{-1}T\}^\dagger$, endow $k\{r^{-1}T\}^\dagger/I$ the quotient seminorm. By the previous lemma, the quotient seminorm is a norm and hence $k\{r^{-1}T\}^\dagger/I$ is a normed k-algebra which is a dense k-subalgebra of the k-Banach algebra $k\{r^{-1}T\}/k\{r^{-1}T\}I$.*

Lemma 2.1.12. *A k-algebra of the form $k\{r^{-1}T\}^\dagger/I$ is a finitely generated weakly complete with respect to the quotient norm.*

Proof. To begin with, we verify that $A := k\{r^{-1}T\}^\dagger/I = k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}^\dagger/I$ is weakly complete. Take an integer $m \in \mathbb{N}$ and elements $f = (f_1, \ldots, f_m) \in (A \backslash \{0\})^m$, a parametre $\delta \in (0,1)$, and a powerseries $F = \sum_{I \in \mathbb{N}^m} F_I T^I \in k\{\|f\|^{-1}T\} = k\{\|f_1\|^{-1}T_1, \ldots, \|f_m\| T_m\}$ such that $|F_I| \|f\|^I < \delta^{|I|}$ for each $I \in \mathbb{N}^m$. Fix a representative

$$\underline{f_i} = \sum_{J \in \mathbb{N}^n} \underline{f_i}_J T^J = \sum_{J \in \mathbb{N}^n} \underline{f_i}_J T_1^{J_1} \cdots T_n^{J_n} \in k\{r^{-1}T\}$$

of f_i such that $\|f_i\| \leq \|\underline{f_i}\| < \delta^{-1/2} \|f_i\|$ for each $i = 1, \ldots, m$. Then one has

$$\begin{aligned}
\sum_{M \leq |I| \leq M'} F_I \underline{f}^I &= \sum_{M \leq |I| \leq M'} F_I \prod_{i=1}^m \left(\sum_{J \in \mathbb{N}^n} \underline{f_i}_J T^J \right)^{I_i} \\
&= \sum_{M \leq |I| \leq M'} F_I \sum_{\mathscr{J} \in (\mathbb{N}^n)^m} \prod_{i=1}^m \prod_{j=1}^{I_i} \underline{f_i}_{\mathscr{J}_{i,j}} \prod_{j=1}^n T_j^{\mathscr{J}_{1,j} + \cdots + \mathscr{J}_{m,j}} \\
&= \sum_{J \in \mathbb{N}^n} \left(\sum_{M \leq |I| \leq M'} F_I \sum_{\mathscr{J}_1 + \cdots + \mathscr{J}_m = J} \prod_{i=1}^m \prod_{j=1}^{I_i} \underline{f_i}_{\mathscr{J}_{i,j}} \right) T^J \\
\left\| \sum_{M \leq |I| \leq M'} F_I \underline{f}^I \right\| &= \left\| \sum_{J \in \mathbb{N}^n} \left(\sum_{M \leq |I| \leq M'} F_I \sum_{\mathscr{J}_1 + \cdots + \mathscr{J}_m = J} \prod_{i=1}^m \prod_{j=1}^{I_i} \underline{f_i}_{\mathscr{J}_{i,j}} \right) T^J \right\| \\
&\leq \sup_{J \in \mathbb{N}^n} \max_{M \leq |I| \leq M'} \max_{\mathscr{J}_1 + \cdots + \mathscr{J}_m = J} |F_I| \prod_{i=1}^m \prod_{j=1}^{I_i} |\underline{f_i}_{\mathscr{J}_{i,j}}| r^J \\
&\leq \max_{M \leq |I| \leq M'} \sup_{J \in \mathbb{N}^n} \max_{\mathscr{J}_1 + \cdots + \mathscr{J}_m = J} |F_I| \left(\prod_{i=1}^m \left(\prod_{j=1}^{I_i} |\underline{f_i}_{\mathscr{J}_{i,j}}| \right) r^{\mathscr{J}_i} \right) \\
&\leq \max_{M \leq |I| \leq M'} \sup_{J \in \mathbb{N}^n} \max_{\mathscr{J}_1 + \cdots + \mathscr{J}_m = J} |F_I| \prod_{i=1}^m \left(\prod_{j=1}^{I_i} |\underline{f_i}_{\mathscr{J}_{i,j}}| r^{\mathscr{J}_{i,j}} \right) \\
&\leq \max_{M \leq |I| \leq M'} |F_I| \prod_{i=1}^m \|\underline{f_i}\|^{I_i} \leq \max_{M \leq |I| \leq M'} |F_I| \prod_{i=1}^m \delta^{-I_i/2} \|f_i\|^{I_i} \\
&< \max_{M \leq |I| \leq M'} \delta^{|I|/2} = \delta^{M/2} \\
&\xrightarrow{M, M' \to \infty} 0
\end{aligned}$$

and hence the infinite sum $\sum_{I \in \mathbb{N}^n} F_I \underline{f}^I$ converges in $k\{r^{-1}T\}$. Now the infinite sum is presented as

$$\sum_{I \in \mathbb{N}^n} F_I \underline{f}^I = \sum_{J \in \mathbb{N}^n} \left(\sum_{I \in \mathbb{N}^n} F_I \sum_{\mathscr{J}_1 + \cdots + \mathscr{J}_m = J} \prod_{i=1}^m \prod_{j=1}^{I_i} \underline{f_i}_{\mathscr{J}_{i,j}} \right) T^J.$$

Take parametres $\delta' \in (0,1)$ such that $\underline{f_i} \in k\{\delta'^2 r^{-1} T\} \subset k\{r^{-1}T\}^\dagger$ and the norm $\|\underline{f_i}\|'$ of $\underline{f_i}$ in $k\{\delta'^2 r^{-1} T\}$ satisfies $\|\underline{f_i}\|' < \delta^{-1}\|f_i\|$ for any $i = 1, \ldots, m$. One obtains

$$\begin{aligned}
& \lim_{|J|\to\infty} \left\| \sum_{I\in\mathbb{N}^n} F_I \sum_{\mathscr{J}_1+\cdots+\mathscr{J}_m=J} \prod_{i=1}^{m} \prod_{j=1}^{I_i} \underline{f_i}_{\mathscr{J}_{i,j}} \right\| \delta'^{-|J|} r^J \\
\leq \ & \lim_{|J|\to\infty} \max_{I\in\mathbb{N}^n} |F_I| \max_{\mathscr{J}_1+\cdots+\mathscr{J}_m=J} \left(\prod_{i=1}^{m} \prod_{j=1}^{I_i} |\underline{f_i}_{\mathscr{J}_{i,j}}| \right) \delta'^{|J|-2|J|} r^J \\
\leq \ & \lim_{|J|\to\infty} \delta'^{|J|} \max_{I\in\mathbb{N}^n} |F_I| \prod_{i=1}^{m} \prod_{j=1}^{I_i} \max_{\mathscr{J}_1+\cdots+\mathscr{J}_m=J} \left(|\underline{f_i}_{\mathscr{J}_{i,j}}| \delta'^{-2|\mathscr{J}_i|} r^{\mathscr{J}_i} \right) \\
\leq \ & \lim_{|J|\to\infty} \delta'^{|J|} \max_{I\in\mathbb{N}^n} |F_I| \prod_{i=1}^{m} \|\underline{f_i}\|'^{I_i} < \lim_{|J|\to\infty} \delta'^{|J|} \max_{I\in\mathbb{N}^n} |F_I| \prod_{i=1}^{m} \delta^{-I_i/2} \|f_i\|^{I_i} \\
= \ & \lim_{|J|\to\infty} \delta'^{|J|} \max_{I\in\mathbb{N}^n} \delta^{-|I|/2} |F_I| \prod_{i=1}^{m} \|f_i\|^{I_i} < \lim_{|J|\to\infty} \delta'^{|J|} \max_{I\in\mathbb{N}^n} \delta^{|I|/2} \\
= \ & \lim_{|J|\to\infty} \delta'^{|J|} = 0,
\end{aligned}$$

and hence the infinite sum $\sum_{I\in\mathbb{N}^m} F_I \underline{f}^I$ is contained in $k\{\delta' r^{-1}T\} \subset k\{r^{-1}T\}^\dagger$. By the continuity of the admissible epimorphism $k\{r^{-1}T\}^\dagger \twoheadrightarrow k\{r^{-1}T\}^\dagger/I = A$, it follows that the infinite sum $\sum_{I\in\mathbb{N}^m} F_I f^I$ converges in A.

The weakly complete k-algebra A is finitely generated, because the images of $T_1, \ldots, T_n \in k\{r^{-1}T\}^\dagger$ in A are obviously generators. $\square$

Definition 2.1.13. *A normed k-algebra is said to be a k-dagger algebra if it admits an admissible epimorphism from a normed k-algebra of the form $k\{r^{-1}T\}^\dagger$. By the previous lemma, a k-dagger algebra is a finitely generated weakly complete k-algebra. If we can set $r = (1, \ldots, 1)$, then we say the k-algebra is a strict k-dagger algebra.*

Definition 2.1.14. *Let A be a normed k-algebra, and denote by $\hat{A}$ the completion of A. Define $A^\dagger \subset \hat{A}$ the subset of elements $a \in \hat{A}$ such that there exist an integer $m \in \mathbb{N}$, elements $f = (f_1, \ldots, f_m) \in (A\backslash\{0\})^m$, and a powerseries $F = \sum_{I\in\mathbb{N}^m} F_I T^I \in k\{\|f\|^{-1}T\}^\dagger$ such that $a = \sum_{I\in\mathbb{N}^m} F_I a^I \in hatA$. Call $A^\dagger$ the weak completion of A. Similarly for a seminormed k-algebra A, define its weak completion as the weak completion of the normed k-algebra A/I_A endowed with the quotient norm with respect to the quotient by the support ideal $I_A \coloneqq \{a \in A \mid \|a\| = 0\}$.*

Lemma 2.1.15. *The subset $A^\dagger \subset \hat{A}$ is a weakly complete k-subalgebra with respect to the restriction of the norm of A, and A is dense in $A^\dagger$. Moreover, $A^\dagger \subset \hat{A}$ is the minimal weakly complete k-subalgebra containing A, which is the intersection of all weakly complete k-subalgebra of $\hat{A}$ containing A.*

Lemma 2.1.16. *For a bounded k-algebra homomorphism $\phi\colon A \to B$ between normed k-algebras, denote by $\hat{\phi}\colon \hat{A} \to \hat{B}$ the unique bounded extension of ϕ. Then one has*

$\hat{\phi}(A^\dagger) \subset B^\dagger$. *The restriction* $\phi^\dagger := \hat{\phi}|_{A^\dagger}^{B^\dagger} \colon A^\dagger \to B^\dagger$ *of* $\hat{\phi}$ *is the unique bounded extension of* ϕ.

Lemma 2.1.17. *The completion of an weakly complete k-algebra is a k-affinoid algebra.*

Definition 2.1.18. *Denote by* (k-*NAlg*) *and* (k-*WBanach*) *the categories of normed k-algebras and weakly complete k-algebras respectively whose morphisms are bounded k-algebra homomorphism.*

Definition 2.1.19. *Define the functor* $\dagger \colon (k\text{-}NAlg) \to (k\text{-}WBanach)$ *by the correspondences* $A \rightsquigarrow A^\dagger$ *and* $(\phi \colon A \to B) \rightsquigarrow (\phi^\dagger \colon A^\dagger \to B^\dagger)$*, and call it the weak completion functor or the dagger functor.*

Lemma 2.1.20. *The completion functor* $\hat{(\cdot)} \colon (k\text{-}Alg) \to (k\text{-}Banach)$ *uniquely factors the weak completion fanctor* $\dagger \colon (k\text{-}NAlg) \to (k\text{-}WBanach)$*. Denote also by* $\hat{(\cdot)}$ *the unique extension* $(k\text{-}WBanach) \to (k\text{-}Banach)$ *of* $\hat{(\cdot)} \colon (k\text{-}Alg) \to (k\text{-}Banach)$*, and call it the completion functor.*

Lemma 2.1.21. *The weak completion functor* $\dagger \colon (k\text{-}NAlg) \to (k\text{-}WBanach)$ *and the completion functor* $\hat{(\cdot)} \colon (k\text{-}WBanach) \to (k\text{-}Banach)$ *are the left adjoints of the fully faithful forgetful functors* $(k\text{-}WBanach) \to (k\text{-}NAlg)$ *and* $(k\text{-}Banach) \to (k\text{-}WBanach)$ *respectively. In other words, one has the canonical functorial bijective maps*

$$\begin{array}{rcl} Hom_{(k\text{-}NAlg)}(A, B) & = & Hom_{(k\text{-}WBanach)}(A^\dagger, B) \\ \text{and } \; Hom_{(k\text{-}WBanach)}(B, C) & = & Hom_{(k\text{-}Banach)}(\hat{B}, C) \end{array}$$

for a normed k-algebra A, an weakly complete k-algebra B, and a k-Banach algebra C.

Definition 2.1.22. *For an weakly complete k-algebra A, denote by* $\mathscr{M}(A)$ *the set of bounded multiplivative seminorms of A, and endow it the weakest topology with respect to which the map* $|f| \colon \mathscr{M}(A) \to [0, \infty) \colon t \mapsto |f(t)| := t(f)$ *is continuous for any element* $f \in A$*. For a bounded k-algebra homomorphism* $\phi \colon A \to B$ *between weakly complete k-algebras A and B, the set-theoretical map*

$$\begin{array}{rcl} \mathscr{M}(\phi) = \phi^* \colon \mathscr{M}(B) & \to & \mathscr{M}(A) \\ t & \mapsto & (t \circ \phi \colon f \mapsto t(\phi(f))) \end{array}$$

is a continuous map. These correspondence determines the functor

$$\mathscr{M} \colon (k\text{-}WBanach) \to (Top).$$

For an weakly complete k-algebra A, the canonical isometric embedding $A \hookrightarrow \hat{A}$ *induces the canonical homeomorphism* $\mathscr{M}(\hat{A}) \to \mathscr{M}(A)$*, and hence identify* $\mathscr{M}(\hat{A}) = \mathscr{M}(A)$*. Note that the functor* $\mathscr{M} \colon (k\text{-}WBanach) \to (Top)$ *coincides with the composition of the analytification functor* $\hat{(\cdot)} \colon (k\text{-}WBanach) \to (k\text{-}Banach)$ *and Berkovich's spectrum functor* $\mathscr{M} \colon (k\text{-}Banach) \to (Top)$.

We verify that some basic operations for seminormed algebras preserve the weak completeness and that a universal property holds if one replaces the corresponding functor from the category of seminormed algebras to the composition of the forgetful functor, the corresponding functor, and the weak completion functor.

Lemma 2.1.23 (subalgebra)**.** *Let A be an weakly complete seminormed k-algebra, and $B \subset A$ a closed k-subalgebra. Then B is weakly complete with respect to the restriction of the seminorm of A.*

Proof. Obviously the topological condition (ii) is inherited by a closed k-subalgebra. □

Lemma 2.1.24 (quotient)**.** *Let A be an weakly complete seminormed k-algebra, and $J \subsetneq A$ a proper ideal. Then the quotient A/J is weakly complete with respect to the quiotient seminorm, and satisfies the universality of the quotient, namely the homomorphism theorem holds in the category* (k*-WBanach*)*.*

Proof. Take an integer $m \in \mathbb{N}$, elements $f = (f_1, \ldots, f_m) \in (A/J \backslash \{0\})^m$, a parametre $\delta \in (0,1)$, and a powerseries $F = \sum_{I \in \mathbb{N}^m} F_I T^I \in k\{\|f\|^{-1} T\} = k\{\|f_1\|^{-1} T_1, \ldots, \|f_m\| T_m\}$ such that $|F_I| \|f\|^I < \delta^{|I|}$ for each $I \in \mathbb{N}^m$. Fix representatives $\underline{f} = (\underline{f}_1, \ldots, \underline{f}_m) \in (A \backslash \{0\})^m$ such that $\|f_i\| \leq \|\underline{f}_i\| < |\delta|^{-1/2} \|f_i\|$. Then one has $|F_I| \|\underline{f}\|^I < |\delta|^{-|I|/2} |F_I| \|f\|^I < \delta^{|I|/2}$ for each $I \in \mathbb{N}^m$. Since A is weakly complete, the infinite sum $\sum_{I \in \mathbb{N}^m} F_I \underline{f}^I$ converges in A. Therefore by the boundedness of the canonical projection $A \to A/J$, the infinite sum $\sum_{I \in \mathbb{N}^m} F_I f^I$ converges in A/J. The universal property is trivial. □

Lemma 2.1.25 (direct limit)**.** *Let $(A_i)_{i \in I}$ be a direct system of weakly complete k-algebras whose transitive maps are contraction maps. Then the direct limit $\varinjlim A_i$ exists in* (k*-WBanach*)*. Its underlying k-algebra is the quotient of the direct limit of the underlying k-algebras by the support of the direct limit seminorm, and its norm is the quotient norm of the direct limit seminorm.*

Proof. It suffices to show that the direct limit of the underlying k-algebra is weakly complete with respect to its direct limit seminorm. It is trivial because any k-algebra homomorphim from a finitely generated k-algebra $k[T]$ fuctors through the weakly complete k-algebra A_i for some $i \in I$. □

Corollary 2.1.26. *A direct system of uniform weakly complete k-algebras is representative in* (k*-WBanach*) *by a uniform weakly complete k-algebra, and a direct system of weakly complete k-algebras whose transitive maps are isometric admits the direct limit whose underlying k-algebra is the direct limit of k-algebras and whose norm is the direct limit seminorm.*

Corollary 2.1.27. *The normed k-algebra $\overline{k}$ is weakly complete.*

Lemma 2.1.28 (direct product)**.** *A family $(A_i)_{i \in I}$ of weakly complete k-algebras admits the direct product $\prod_{i \in I} A_i$ in the category* (k*-WBanach*)*. It is the weakly complete k-subalgebra of the direct product $\prod_{i \in I} \hat{A}_i$ in the category* (k*-Banach*) *consisting of elements whose entries are contained in the algebraic direct product $\prod_{i \in I} |A_i|$ of the underlying k-algebras $|A_i|$.*

Lemma 2.1.29 (tensor product)**.** *For weakly complete k-algebras A_0, A_1, and A_2, and for bounded k-algebra homomorphims $A_0 \to A_1$ and $A_0 \to A_2$, denote by $A_1 \otimes^{\dagger}_{A_0} A_2$ the weak completion of the image of the algebraic tensor product $A_1 \otimes_{A_0} A_2$ in the k-Banach algebra $\hat{A}_1 \hat{\otimes}_{\hat{A}_0} \hat{A}_2$. Then the bounded k-algebra homomorphisms $A_1 \to A_1 \otimes_{A_0} A_2 \to A_1 \otimes^{\dagger}_{A_0} A_2$ and $A_2 \to A_1 \otimes_{A_0} A_2 \to A_1 \otimes^{\dagger}_{A_0} A_2$ satisfy the universality of the colimit in the category* (k*-WBanach*)*. In particular, the operation $\otimes^{\dagger}$ is commutative and associative. Call $A_1 \otimes^{\dagger}_{A_0} A_2$ the weakly complete tensor product of A_1 and A_2 over A_0.*

Proof. Trivial by the universalities of the algebraic tensor product and the weak completion functor. □

Corollary 2.1.30 (direct sum)**.** *For weakly complete k-algebras $A_1, \ldots, A_m$, the weakly complete tensor product $A_1 \otimes^{\dagger}_k \cdots \otimes^{\dagger}_k A_m$ satisfies the universality of the direct sum in the category* (k*-WBanach*)*.*

Similar with affinoid algebras, dagger algebras are used to construct a geometric object called a dagger space. Since the completion functor sends a k-dagger algebra to a k-affinoid algebra, one obtains the analytification functor from the category of dagger spaces to the category of analytic spaces. Conversely, there is a canonical functor from the category of algebraic varieties to the category of dagger spaces. The composition of this functor and the analytification functor coincides with the Berkovich's analytification functor. In this paper, we do not define what is a dagger space and what structure a dagger space possesses. For more detail of a dagger space, see [KLO1].

Definition 2.1.31. *Denote by* (k*-Dg*) *the category of dagger spaces, and by $\hat{(\cdot)} \colon (k\text{-Dg}) \to (k\text{-An})$ the analytification functor.*

Definition 2.1.32. *For a dagger space X, denote by $|X|$ the underlying topological set, by $O^{\dagger}_X$ the overconvergent structure sheaf, and by O_X the completion of the overconvergent structure sheaf, which coincides with the pull-back of the structure sheaf $O_{\hat{X}}$ of the analytification $\hat{X}$.*

2.2 Ring of overconvergent analytic functions

Definition 2.2.1 (ring of overconvergent analytic functions)**.** *Let $S \subset \mathbb{R}^n$ be a polytope. Denote by $k^{\dagger}_S$ the weak completion of $k[E_{k,n}]$ with respect to the Gauss seminorm $\|\cdot\|_S$. The weakly complete k-algebra is the minimal weakly complete k-subalgebra of k_S containing the image of $k[E_{k,n}]$ by the definition of the weak completion. Call an element $f \in k^{\dagger}_S$ an overconvergent analytic function on S, or just say f is defined on a neighbourhood of S. Since the image of $k[E_{k,n}]$ in k_S is dense, $k^{\dagger}_S$ is a dense k-subalgebra of k_S.*

Definition 2.2.2. *Formally set $k^{\dagger}_{\emptyset} := 0$ and $\| \cdot \|_{\emptyset} := 0 \colon k^{\dagger}_{\emptyset} \to [0, \infty)$. For each polytope S, associate the unique embedding $\emptyset \hookrightarrow S$ with the bounded k-algebra homomorphism $0 \colon k^{\dagger}_S \to k^{\dagger}_{\emptyset} \colon f \mapsto 0$.*

Be careful about the fact that $k_S^\dagger$ does not coincides with the direct limit

$$\varinjlim_{S \subset \mathrm{Int}(T)} k_T \subset k_S$$

of the rings k_T of analytic functions on a polytope T such that $S \subset \mathrm{Int}(T)$, because an element $x \in E_{k,n}$ such that $|x^{(1)}(1)| = \cdots = |x^{(n)}(1)| = 1$ is a bounded analytic function on $\mathbb{R}^n$, which has no analogous holomorphic function on $\mathbb{C}^n$ by Liouville's theorem. When S is thick, then an element of the weakly complete k-algebra $k_S^\dagger$ is uniquely presented by its coefficients as an element of k_S through the embedding $k_S \hookrightarrow k^{E_{k,n}}$. Similar with k_S, the image of $k_S^\dagger$ is discribed by a certain condition about limit.

Proposition 2.2.3. *Let $S \subset \mathbb{R}^n$ be a thick polytope. Then the image of $k_S^\dagger$ by the embedding $k_S \hookrightarrow k^{E_{k,n}}$ coincides with the k-vector subspace of sequences $(f_x)_{x \in E_{k,n}} \in k^{E_{k,n}}$ satisfying that there exist an integer $m \in \mathbb{N}$, elements $x = (x_1, \ldots, x_m) \in E_{k,n}^m$, and an overconvergent power series*

$$F = \sum_{I \in \mathbb{N}^m} F_I T^I \in k\{\|x\|_S^{-1} T\}^\dagger = k\{\|x_1\|_S^{-1} T_1, \ldots, \|x_m\|_S^{-1} T_m\}^\dagger$$

such that $f_y = 0$ for any $y \in E_{k,n} \backslash x_1^{\mathbb{N}} + \cdots + x_m^{\mathbb{N}}$ and

$$f_y = \sum_{\substack{I \in \mathbb{N}^m \\ y = x^I}} F_I$$

for any $y \in x_1^{\mathbb{N}} + \cdots + x_m^{\mathbb{N}}$, or in other words, f is presented as

$$f = \sum_{I \in \mathbb{N}^>} F_I x^I.$$

Proof. Take an overconvergent analytic funtion $f \in k_S^\dagger$. By the definition of the weak completion, there exist an integer $m' \in \mathbb{N}$, elements $g = (g_1, \ldots, g_{m'}) \in (k[E_{k,n}] \backslash \{0\})^{m'} \subset (k_S \backslash \{0\})^{m'}$, and an overconvergent power series

$$G = \sum_{J \in \mathbb{N}^{m'}} G_J T^J \in k\{\|g\|_S^{-1} T\}^\dagger = k\{\|g_1\|_S^{-1} T_1, \ldots, \|g_{m'}\|_S^{-1} T_m\}^\dagger$$

such that $f = \sum_{J \in \mathbb{N}^{m'}} G_J g^J$. Let $\Sigma \sum E_{k,n}$ be the set of all elements $y \in E_{k,n}$ such that $g_{j,y} \neq 0$ for some $j = 1, \ldots, m'$. Since $g_1, \ldots, g_{m'} \in k[E_{k,n}]$, all but finitely many entries of their coefficients are 0, and hence Σ is a finite set. Set $m := \#\Sigma$, and fix a bijective map $x = (x_1, \ldots, x_m) \colon \{1, \ldots, m\} \to \Sigma \colon i \mapsto x_i$. Then one has $g_j = g_{j,1} x_1 + \cdots + g_{j,m} x_m$ for any $j = 1, \ldots, m'$, and

$$f = \sum_{J \in \mathbb{N}^{m'}} G_J g^J = \sum_{J \in \mathbb{N}^{m'}} G_J \prod_{j=1}^{m'} \left(\sum_{i=1}^{m} g_{j,i} x_i \right)^{J_j}$$

$$
\begin{aligned}
&= \sum_{J \in \mathbb{N}^{m'}} G_J \prod_{j=1}^{m'} \sum_{I_1 + \cdots + I_m = J_i} \frac{J_i!}{I_1! \cdots I_m!} \prod_{i=1}^{m} g_{j,i}^{I_i} x_i^{I_i} \\
&= \sum_{\mathscr{I} \in (\mathbb{N}^m)^{m'}} G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}} \prod_{j=1}^{m'} \left(\frac{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})!}{\mathscr{I}_{j,1}! \cdots \mathscr{I}_{j,m}!} \prod_{i=1}^{m} g_{j,i}^{\mathscr{I}_{j,i}} x_i^{\mathscr{I}_{j,i}} \right) \\
&= \sum_{I \in \mathbb{N}^m} \left(\sum_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}} \prod_{j=1}^{m'} \left(\frac{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})!}{\mathscr{I}_{j,1}! \cdots \mathscr{I}_{j,m}!} \prod_{i=1}^{m} g_{j,i}^{\mathscr{I}_{j,i}} \right) \prod_{i=1}^{m} x_i^{I_i} \right) \\
&= \sum_{I \in \mathbb{N}^m} \left(\sum_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}} \frac{\prod_{j=1}^{m'} (\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})!}{\prod_{i=1}^{m} \prod_{j=1}^{m'} \mathscr{I}_{j,i}!} \left(\prod_{i=1}^{m} \prod_{j=1}^{m'} g_{j,i}^{\mathscr{I}_{j,i}} \right) \right) x^I.
\end{aligned}
$$

Note that the sum in the big blacket in the last equality is a finite sum. Consider the formal power series

$$
\begin{aligned}
F &= \sum_{I \in \mathbb{N}^m} F_I T^I \\
&:= \sum_{I \in \mathbb{N}^m} \left(\sum_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}} \frac{\prod_{j=1}^{m'} (\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})!}{\prod_{i=1}^{m} \prod_{j=1}^{m'} \mathscr{I}_{j,i}!} \left(\prod_{i=1}^{m} \prod_{j=1}^{m'} g_{j,i}^{\mathscr{I}_{j,i}} \right) \right) T^I \\
&\in k[[T]].
\end{aligned}
$$

It suffices to show that $F \in k\{\|x\|_S^{-1} T\}^\dagger = k\{\|x_1\|_S^{-1} T_1, \ldots, \|x_m\|_S^{-1} T_m\}^\dagger$. Take a parametre $\delta \in (0,1)$ such that $G \in k\{\delta \|g\|^{-1} T\} \subset k\{\|g\|^{-1} T\}^\dagger$. Then one has

$$
\begin{aligned}
&|F_I| \delta^{-|I|} \|x\|^I \\
&= \left| \sum_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}} \frac{\prod_{j=1}^{m'} (\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})!}{\prod_{i=1}^{m} \prod_{j=1}^{m'} \mathscr{I}_{j,i}!} \left(\prod_{i=1}^{m} \prod_{j=1}^{m'} g_{j,i}^{\mathscr{I}_{j,i}} \right) \right| \delta^{-|I|} \|x\|^I \\
&\leq \max_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} |G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}}| \left| \frac{\prod_{j=1}^{m'} (\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})!}{\prod_{i=1}^{m} \prod_{j=1}^{m'} \mathscr{I}_{j,i}!} \right| \left(\prod_{i=1}^{m} \prod_{j=1}^{m'} |g_{j,i}|^{\mathscr{I}_{j,i}} \right) \delta^{-|I|} \|x\|^I \\
&\leq \max_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} |G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}}| \delta^{-|I|} \prod_{i=1}^{m} \prod_{j=1}^{m'} \left(|g_{j,i}|^{\mathscr{I}_{j,i}} \|x_i\|^{\mathscr{I}_{j,i}} \right) \\
&\leq \max_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} |G_{(\mathscr{I}_{j,1} + \cdots + \mathscr{I}_{j,m})_{j=1}^{m}}| \delta^{-|I|} \prod_{i=1}^{m} \prod_{j=1}^{m'} \|g_j\|_S^{\mathscr{I}_{j,i}}
\end{aligned}
$$

$$= \max_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} |G_{(\mathscr{I}_{j,1}+\cdots+\mathscr{I}_{j,m})_{j=1}^m}| \delta^{-|I|} \prod_{j=1}^{m'} \|g_j\|_S^{\mathscr{I}_{j,1}+\cdots+\mathscr{I}_{j,m}}$$

$$= \max_{\substack{\mathscr{I} \in (\mathbb{N}^m)^{m'} \\ \mathscr{I}_1 + \cdots + \mathscr{I}_{m'} = I}} |G_{(\mathscr{I}_{j,1}+\cdots+\mathscr{I}_{j,m})_{j=1}^m}| \delta^{-|I|} \prod_{j=1}^{m'} \|g\|_S^{(\mathscr{I}_{j,1}+\cdots+\mathscr{I}_{j,m}} = \sup_{|J| \geq |I|} |G_J| \|g\|_S^J$$

$$\stackrel{|I| \to \infty}{\longrightarrow} 0,$$

and it implies that $F \in k\{\|x\|_S^{-1} T\}^\dagger$. $\square$

Corollary 2.2.4. *Let $S \subset \mathbb{R}^n$ be a polytope. For an overconvergent analytic function $f \in k_S^\dagger$, there exist an integer $m \in \mathbb{N}$, elements $x = (x_1, \ldots, x_m) \in E_{k,n}^m$, and an overconvergent power series*

$$F = \sum_{I \in \mathbb{N}^m} F_I T^I \in k\{\|x\|'^{-1}_S T\}^\dagger = k\{\|x_1\|'^{-1}_S T_1, \ldots, \|x_m\|'^{-1}_S T_m\}^\dagger$$

such that f is presented as the limit of the image of the sum

$$\sum_{|I| < M} F_I x^I \in k[E_{k,n}]$$

by the canonical k-algebra homomorphism $k[E_{k,n}] \to k_S^\dagger$ with respect to the upper bound $M \to \infty$.

Proof. We have already verified it in the case S is thick. In general, take a thick polytope $T \subset \mathbb{R}^m$ and an isomorphic integral affine map $a \colon S \to T$, and let $(A, b) \in \mathrm{M}(m, n; \mathbb{Z}) \times \mathbb{Z}^m$ be its presentation. Then $a^*(x) = x(b) x^A$, $\|x\|_T = \|a^*(x)\|_S \leq |x(b)| \|x^A\|'_S$, $x(b) \in k$, and $x^A \in E_{k,n}$ for any $x \in E_{k,m}$. For an overconvergent analytic function $f \in k_S^\dagger$, present $(a^*)^{-1}(f) = \sum_{I \in \mathbb{N}^l} F_I x^I$ by an integer $l \in \mathbb{N}$, elements $x = (x_1, \ldots, x_l) \in E_{k,m}^l$, and an overconvergent power series

$$F = \sum_{I \in \mathbb{N}^l} F_I T^I \in k\{\|x\|_T^{-1} T\}^\dagger = k\{\|x_1\|_T^{-1} T_1, \ldots, \|x_l\|_T^{-1} T_l\}^\dagger.$$

Set

$$G := \sum_{I \in \mathbb{N}^l} F_I x^I(b) T^I \in k[[T]] = k[[T_1, \ldots, T_l]].$$

Then obviously one has $G \in k\{\|x^A\|'^{-1}_S T\}^\dagger$ by the inequality $\|x\|_T \leq |x(b)| \|x^A\|'_S$ and $G(x^A) = f$. $\square$

In this section, we just replace k_S to $k_S^\dagger$ and repeat the totally same process in §1. Therefore we otfen omit the proof of the corresponding proposition of what we have already verified in the case of k_S.

Proposition 2.2.5. *Let $a\colon S \to T$ be an integral affine map between polytopes S and T. Then the associated bounded k-algebra homomorphism $a^*\colon k_T \to k_S$ sends $k_T^\dagger$ to $k_S^\dagger$. Therefore it induces the bounded k-algebra homomorphism $a^*\colon k_T^\dagger \to k_S^\dagger$. In particular for polytopes $S \leq T$, the restriction map $k_T \to k_S$ induces the injective bounded k-algebra homomorphism $k_T^\dagger \to k_S^\dagger$.*

Proof. Trivial. A bounded k-algebra homorphism $k_{T'} \to k_T$ between the k-Banach algebras associated with polytopes $T \subset \mathbb{R}^m$ and $T' \subset \mathbb{R}^{m'}$ which is a unique bounded extension of a k-algebra homomorphism $k[E_{k,m'}] \to k[E_{k,m}]$ always sends $k_{T'}^\dagger$ to $k_T^\dagger$. □

Corollary 2.2.6. *If k is algebraically closed, then an affine map $a\colon S \to T$ induces the bounded k-algebra homomorphism $a^*\colon k_T^\dagger \to k_S^\dagger$ which is a contraction map in the similar way.*

Corollary 2.2.7. *Let $a\colon S \to T$ be an isomorphic integral affine map. Then $a^*\colon k_T^\dagger \to k_S^\dagger$ is an isometric isomorphism. If k is algebraically closed, the same holds for an isomorphic affine map.*

Corollary 2.2.8. *Let $S \subset \mathbb{R}^n$ be a polytope. Then there exist the unique integer $m \leq n \in \mathbb{N}$ and a thick polytope $T \subset \mathbb{R}^m$ such that $k_S^\dagger$ is isometrically isomorphic to $k_T^\dagger$ through the isomorphism associated with an isomorphic integral affine map $S \to T$.*

Proposition 2.2.9 (ground field extension)**.** *Let S be a polytope, and K/k an extension of complete non-Archimedean fields. The canonical bounded k-algebra homomorphism $k_S^\dagger \otimes_k^\dagger K \to K_S^\dagger$ is an isometric isomorphism onto the image.*

Proof. It is the restriction of the canonical bounded k-algebra homomorphism $k_S \hat{\otimes}_k K \to K_S$, which is an isometric isomorphism onto the image. □

Definition 2.2.10 (fibre product)**.** *For polytopes $S_1, \ldots, S_m$ and $S \coloneqq S_1 \times \cdots \times S_m$, the bounded multiplication $k_{S_1} \hat{\otimes}_k \cdots \hat{\otimes}_k k_{S_m} \to k_S$ determines the bounded multiplication*

$$k_{S_1}^\dagger \otimes_k^\dagger \cdots \otimes_k^\dagger k_{S_m}^\dagger \to k_S^\dagger$$

and gives the non-commutative ring structure of

$$\prod_{n=0}^{\infty} k_{[0,1]^n}^\dagger.$$

Proposition 2.2.11. *For polytopes $S_1, \ldots, S_m$ and $S \coloneqq S_1 \times \cdots \times S_m$, the canonical bounded k-algebra homomorphism $k_{S_1}^\dagger \otimes_k^\dagger \cdots \otimes_k^\dagger k_{S_m}^\dagger \to k_S^\dagger$ is an isometric isomorphism. In particular one has the canonical isometric isomorphism*

$$\prod_{n=0}^{\infty} k_{[0,1]^n}^\dagger \cong_k \prod_{n=0}^{\infty} \left(k_{[0,1]}^\dagger\right)^{\otimes^\dagger n}$$

of non-commutative k-algebras.

Proposition 2.2.12 (Galois representation)**.** *The Galois representaion $K_S \times G_k \to K_S$ sends $K_S^\dagger \times G_k$ to $K_S^\dagger$ for any Galois extension K/k of complete non-Archimedean fields contained in C, and hence it induces the isometric Galois representation $K_S^\dagger \times G_k \to K_S^\dagger$.*

Proof. Trivial. A bounded k-algebra homorphism $K_{T'} \to K_T$ between the k-Banach algebras associated with polytopes $T \subset \mathbb{R}^m$ and $T' \subset \mathbb{R}^{m'}$ which is a unique bounded extension of a k-algebra homomorphism $K[E_{K,m'}] \to K[E_{K,m}]$ always sends $K_{T'}^\dagger$ to $K_T^\dagger$. □

Lemma 2.2.13. *Let $a\colon S \to T$ be an integral affine map between polytopes S and T. Then the associated bounded k-algebra homomorphism $a^*\colon C_T^\dagger \to C_S^\dagger$ is G_k-equivariant.*

Proof. It is the restriction of the G_k-equivariant homomorphism $a^*\colon C_T \to C_S$. □

Proposition 2.2.14. *Suppose k is a local field, i.e. a complete discrete valuation field with finite residue field, and let K/k be a finite extension contained in $\overline{k}$. For a polytope S, the G_K-invariants $(k_S^\dagger)^{G_K}$ of the G_K-representation $k_S^\dagger$ coincides with $k \subset k_S^\dagger$. In particular $(k_S^\dagger)^{G_k} = k$.*

Proof. Just see the inclusion $k \subset (k_S^\dagger)^{G_k} \subset k_S^{G_k} = k$. □

Corollary 2.2.15. *In the same situation above, if K/k is a Galois extension, the G_k-invariants $(K_S^\dagger)^{G_k}$ of the G_k-representation K_S coincides with the k.*

Corollary 2.2.16. *In the same situation above, the G_k-invariants $(k_S^\dagger \otimes_k^\dagger \overline{k})^{G_k}$ of the G_k-representation $k_S^\dagger \otimes_k^\dagger \overline{k}$ coincides with k.*

Proposition 2.2.17. *For a polytope S, the weakly complete k-algebra $k_S^\dagger$ is a uniformly normed integral domain.*

Proof. It is a normed subring of the uniformly normed integral domain k_S. □

Lemma 2.2.18 (involution)**.** *The involution $*\colon k_{[0,m]} \to k_{[0,m]}$ sends $k_{[0,m]}^\dagger$ to $k_{[0,m]}^\dagger$, and hence it induces the isometric isomorphism $*\colon k_{[0,m]}^\dagger \to k_{[0,m]}^\dagger$.*

Proof. The involution $*\colon k_{[0,m]} \to k_{[0,m]}$ sends $k[E_{k,1}]$ to $k[E_{k,1}]$ by the definition. □

2.3 Non-Archimedean dagger realisation of a polytope

Proposition 2.3.1 (universality of a rational domain)**.** *Let S be a polytope, A a k-dagger algebra, $V \subset \mathscr{M}(A)$ a rational domain ([KLO1]), and $\psi\colon A \to k_S^\dagger$ a bounded k-homomorphism. If the image $\psi^*(|S_k|) \subset |\mathscr{M}(A)|$ by the continuous map $\psi^*\colon |S_k| \to |\mathscr{M}(A)|$ associated with $\psi\colon A \to k_S$ is contained in $|V| \subset |\mathscr{M}(A)|$, the homomorphism $\psi\colon A \to k_S$ uniquely factors through the canonical homomorphism $A \to A_V$ of dagger algebras.*

Proposition 2.3.2 (universality of a Weierstrass domain)**.** *Let S be a polytope, A a k-affinoid algebra, $V \subset \mathscr{M}(A)$ a Weierstrass domain ([BER1]), and $\psi \colon A \to k_S$ a bounded k-algebra homomorphism. If the image $\psi^*(|S|) \subset |\mathscr{M}(A)|$ of the underlying polytope $S \subset S_k$ by the continuous map $\psi^* \colon |S_k| \to |\mathscr{M}(A)|$ associated with $\psi \colon A \to k_S$ is contained in $|V| \subset |\mathscr{M}(A)|$, the homomorphism $\psi \colon A \to k_S$ uniquely factors through the canonical homomorphism $A \to A_V$ of affinoid algebras, and hence the image $\psi^*(|S_k|) \subset |\mathscr{M}(A)|$ is contained in $|V| \subset |\mathscr{M}(A)|$.*

Proposition 2.3.3 (Tate's acyclicity)**.** *Let $S, S_1, \ldots, S_m$ be polytopes such that $S = S_1 \veebar \cdots \veebar S_m$. Then the restriction maps $k_S^\dagger \hookrightarrow k_{S_i}^\dagger$ induce the admissible exact sequence*

$$0 \to k_S^\dagger \to \prod_{i=1}^m k_{S_i}^\dagger \to \prod_{i,j=1}^m k_{S_i \wedge S_j}^\dagger$$

of k-Banach algebras.

Proof. These propositions are easily verified in the totally same way in the proofs of Proposition 1.2.17, Proposition 1.2.19, and Proposition 1.2.29. Note that we used Banach's open mapping theprem in the proof of the admissibility of the ecat sequence in Tate's acyclicity, Proposition 1.2.29, but one does not have to extend Banach's open mapping theorem to weakly complete k-algebras. The sequence

$$0 \to k_S^\dagger \to \prod_{i=1}^m k_{S_i}^\dagger \to \prod_{i,j=1}^m k_{S_i \wedge S_j}^\dagger$$

is isometrically embedded in the admissible exact sequence

$$0 \to k_S \to \prod_{i=1}^m k_{S_i} \to \prod_{i,j=1}^m k_{S_i \wedge S_j}.$$

□

Corollary 2.3.4. *Let $S, S_1, \ldots, S_m$ be polytopes such that $S = S_1 \veebar \cdots \veebar S_m$, and A a weakly complete k-algebra or a k-algebra. The exact sequence of Tate's acyclicity, Proposition 2.3.3, induces the set-theoretical exact sequence*

$$* \to Hom_{(k\text{-}WBanach)}(A, k_S) \to \prod_{i=1}^m Hom_{(k\text{-}WBanach)}(A, k_{S_i}) \rightrightarrows \prod_{i,j=1}^m Hom_{(k\text{-}WBanach)}(A, k_{S_i \wedge S_j})$$

or $$* \to Hom_{(k\text{-}Alg)}(A, k_S) \to \prod_{i=1}^m Hom_{(k\text{-}Alg)}(A, k_{S_i}) \rightrightarrows \prod_{i,j=1}^m Hom_{(k\text{-}Alg)}(A, k_{S_i \wedge S_j})$$

respectively.

Definition 2.3.5. *Let S be a polytope. Endow S_k the overconvergent structure presheaf*

$$\begin{array}{rcl} O_S^\dagger \colon (S_k, \tau_S, Cov_S) & \to & (k\text{-}WBanach) \\ T_k & \rightsquigarrow & H^0(T_k, O_S^\dagger) := k_T \\ (i \colon T_k \hookrightarrow T'_k) & \rightsquigarrow & (i^* \colon k_{T'}^\dagger \to k_T^\dagger), \end{array}$$

which is a sheaf of weakly complete k-algebras by Tate's acyclicity, Proposition 2.3.3. Denote also by S_k the G-ringed space $(S_k, \tau_S, Cov_S, O_S^\dagger)$ endowed with the overconvergent structure sheaf $O_S^\dagger$ of k-Banach algebras on the G-topology, and call it the dagger spectrum of $k_S^\dagger$ or the non-Archimedean dagger realisation of S.

Definition 2.3.6. *A k-dagger simplex is the dagger spectrum S_k for some polytope S.*

2.4 Analytic path from a dagger simplex

We define a morphism from a dagger simplex to a dagger space in the same way as we did in §1.4. The categorical general nonsense in §1.3 works, and we omit the corresponding straightforward proofs and explanations to avoid annoying repetitions from §1.4.

Definition 2.4.1. *Denote by $\mathscr{A}_k^\dagger$ the family of all k-dagger algebras of the form $k\{T_1, \ldots, T_n\}^\dagger / I$ for an integer $n \in \mathbb{N}$ and a proper ideal $I \subsetneq k\{T_1, \ldots, T_n\}^\dagger$. Note that $\mathscr{A}_k^\dagger$ is not a proper class in the sense of Von Neumann-Bernays-Gödel set theory, and represents all isomorphic classes of k-dagger algebras.*

Definition 2.4.2. *Let S_k be a k-dagger simplex and X a k-dagger space. Set*

$$Hom(S, \mathscr{A}_k^\dagger, X) := \bigsqcup_{A \in \mathscr{A}_k^\dagger} Hom_{(k\text{-}WBanach)}(A, k_S^\dagger) \times Hom_{(k\text{-}Dg)}(\mathscr{M}(A), X).$$

For an element $\gamma \in Hom(S, \mathscr{A}_k^\dagger, X)$, let $A_\gamma \in \mathscr{A}_k^\dagger$ be the unique k-affinoid algebra such that

$$\gamma \in Hom_{(k\text{-}WBanach)}(A_\gamma, k_S^\dagger) \times Hom_{(k\text{-}Dg)}(\mathscr{M}(A_\gamma), X).$$

Denote by $\gamma^{(0)} \colon k_S^\dagger \to A_\gamma$ and $\gamma^{(1)} \colon \mathscr{M}(A_\gamma) \to X$ the unique bounded k-algebra homomorphism and the unique morphism such that

$$\gamma = (\gamma^{(0)}, \gamma^{(1)}) \in Hom_{(k\text{-}WBanach)}(A_\gamma, k_S^\dagger) \times Hom_{(k\text{-}Dg)}(\mathscr{M}(A_\gamma), X).$$

Definition 2.4.3. *For two elements $\gamma_0, \gamma_1 \in Hom(S, \mathscr{A}_k^\dagger, X)$, we write $\gamma_0 \sim \gamma_1$ if there exist an element $\gamma_{0.5} \in Hom(S, \mathscr{A}_k^\dagger, X)$ and bounded k-algebra homomorphisms $\phi_{0.5,0} \colon A_{\gamma_0} \to$*

$A_{\gamma_{0.5}}$ and $\phi_{0.5,1}\colon A_{\gamma_1} \to A_{\gamma_{0.5}}$ such that the diagrams

$$
\begin{tikzcd}
k_S^\dagger \arrow[d, equal] & A_{\gamma_0} \arrow[l, "\gamma_0^{(0)}"'] \arrow[d, "\phi_{0.5,0}"] \\
k_S^\dagger \arrow[d, equal] & A_{\gamma_{0.5}} \arrow[l, "\gamma_{0.5}^{(0)}"'] \\
k_S^\dagger & A_{\gamma_1} \arrow[l, "\gamma_1^{(0)}"'] \arrow[u, "\phi_{0.5,1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{M}(A_{\gamma_0}) \arrow[r, "\gamma_0^{(1)}"] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_{0.5}}) \arrow[r, "\gamma_{0.5}^{(1)}"] \arrow[u, "\mathscr{M}(\phi_{0.5,0})"] \arrow[d, "\mathscr{M}(\phi_{0.5,1})"'] & X \arrow[d, equal] \\
\mathscr{M}(A_{\gamma_1}) \arrow[r, "\gamma_1^{(1)}"] & X
\end{tikzcd}
$$

commute.

Lemma 2.4.4. *The binary relation $\sim$ on $Hom(S, \mathscr{A}_k^\dagger, X)$ is an equivalence relation.*

Definition 2.4.5. *Set $Hom(S, X) := Hom(S, \mathscr{A}_k^\dagger, X)/\sim$. Call an element $\gamma \in Hom(S, X)$ a morphism, and write $\gamma\colon S \to X$.*

Definition 2.4.6. *When S is the cube $[0,1]^n$, the standerd simplex*

$$\Delta^n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_0 + \cdots + t_n = 1, 0 \leq t_i \leq 1, {}^\forall i = 0, \ldots, n \right\},$$

or something like them, then call a morphism from S a dagger path.

Recall that one has analytification functors

$$\hat{(\cdot)}\colon (k\text{-WBanach}) \to (k\text{-Banach})$$

and

$$\hat{(\cdot)}\colon (k\text{-Dg}) \to (k\text{-An}).$$

We verify they induce the analytification map

$$\hat{(\cdot)}\colon \mathrm{Hom}(S, X) \to \mathrm{Hom}(S, \hat{X}).$$

Proposition 2.4.7 (analytification)**.** *For an element $\gamma \in Hom(S, \mathscr{A}_k^\dagger, X)$, set*

$$\hat{\gamma} := (\widehat{\gamma^{(0)}}, \widehat{\gamma^{(1)}}) \in Hom_{(k\text{-Banach})}(\hat{A}_\gamma, k_S) \times Hom_{(k\text{-An})}(\hat{\mathscr{A}}, \hat{X}) \subset Hom(S, \mathscr{A}_k, \hat{X})$$

This correspondence

$$
\begin{array}{rccc}
\hat{(\cdot)}\colon & Hom(S, \mathscr{A}_k^\dagger, X) & \to & Hom(S, \mathscr{A}_k, \hat{X}) \\
 & \gamma & \mapsto & \hat{\gamma}
\end{array}
$$

is invariant under the equivalence relation $\sim$, and hence it determines a set-theoretical map

$$\hat{(\cdot)}\colon Hom(S, X) \;\to\; Hom(S, \hat{X}).$$

Call it the analytification map.

Now we repete the corresponding propositions in §1.4 just replacing the terms "an affinoid simplex", "an affinoid space", and "an analytic space" to the terms "a dagger simplex", "an affinoid dagger space", and "a dagger space". Each proposition is verified in the totaly same ways.

Lemma 2.4.8. *For an element $\gamma \in Hom(S, \mathscr{A}_k^\dagger, X)$, set*

$$H^0(-,\mathbb{G}_a^\dagger)(\gamma) \coloneqq \gamma^{(0)} \circ H^0(-,\mathbb{G}_a^\dagger)(\gamma^{(1)}) \colon H^0(X, O_X^\dagger) \xrightarrow{H^0(-,\mathbb{G}_a^\dagger)(\gamma^{(1)})} A_\gamma \xrightarrow{\gamma^{(0)}} k_S^\dagger.$$

This correspondence

$$\begin{array}{rcl} H^0(-,\mathbb{G}_a^\dagger)\colon Hom(S, \mathscr{A}_k^\dagger, X) & \to & Hom_{(k\text{-}Alg)}(H^0(X, O_X^\dagger), k_S^\dagger) \\ \gamma & \to & H^0(-,\mathbb{G}_a^\dagger)(\gamma) \end{array}$$

is invariant under the equivalence relation $\sim$, and hence it determines a set-theoretical map

$$\begin{array}{rcl} H^0(-,\mathbb{G}_a^\dagger)\colon Hom(S, X) & \to & Hom_{(k\text{-}Alg)}(H^0(X, O_X^\dagger), k_S^\dagger) \\ \gamma & \to & H^0(-,\mathbb{G}_a^\dagger)(\gamma). \end{array}$$

Lemma 2.4.9. *For an element $\gamma \in Hom(S, \mathscr{A}_k^\dagger, X)$, set*

$$\gamma^\sharp \coloneqq \gamma^{(1)\sharp} \circ \mathscr{M}(\gamma^{(0)})^\sharp \colon |S_k| \xrightarrow{\mathscr{M}(\gamma^{(0)})^\sharp} |\mathscr{M}(A_\gamma)| \xrightarrow{\gamma^{(1)\sharp}} |X|.$$

This correspondence

$$\begin{array}{rcl} \sharp\colon Hom(S, \mathscr{A}_k^\dagger, X) & \to & Hom_{(Top)}(|S_k|, |X|) \\ \gamma & \to & \gamma^\sharp \end{array}$$

is invariant under the equivalent relation $\sim$, and hence it determines a set-theoretical map

$$\begin{array}{rcl} \sharp\colon Hom(S, X) & \to & Hom_{(Top)}(|S_k|, |X|) \\ \gamma & \to & \gamma^\sharp. \end{array}$$

Call the image $\gamma^\sharp \colon |S_k| \to |X|$ of a morphism $\gamma \colon S \to X$ the underlying continuous map of γ.

Proposition 2.4.10 (adjoint property)**.** *Let S_k be a k-dagger simplex and $\mathscr{M}(A)$ a k-affinoid dagger space. The canonical map*

$$H^0(-,\mathbb{G}_a^\dagger)\colon Hom(S, \mathscr{M}(A)) \ \to \ Hom_{(k\text{-}Alg)}(A, k_S^\dagger)$$

defined above induces a set-theoretical bijection

$$H^0(-,\mathbb{G}_a^\dagger)\colon Hom(S, \mathscr{M}(A)) \to Hom_{(k\text{-}WBanach)}(A, k_S^\dagger).$$

Denote by

$$\begin{array}{rcl}\mathscr{M}: Hom_{(k\text{-}WBanach)}(A, k_S^\dagger) & \to & Hom(S, \mathscr{M}(A)) \\ \phi & \mapsto & \mathscr{M}(\phi)\end{array}$$

the inverse map.

Proposition 2.4.11. *Let S_k and T_k be k-dagger simplices, and X and Y k-dagger spaces. The maps*

$$\begin{array}{rcl}Hom_{(Polytopes)}(S, T) \times Hom_{(k\text{-}WBanach)}(A, k_T^\dagger) & \to & Hom_{(k\text{-}WBanach)}(A, k_S^\dagger) \\ (a, \phi) & \to & \phi \circ a^*\end{array}$$

and

$$\begin{array}{rcl}Hom_{(k\text{-}Dg)}(\mathscr{M}(A), X) \times Hom_{(k\text{-}An)}(X, Y) & \to & Hom_{(k\text{-}Dg)}(\mathscr{M}(A), Y) \\ (\chi, \psi) & \to & \psi \circ \chi\end{array}$$

for a k-dagger algebra $A \in \mathscr{A}_k^\dagger$ induce the composition map

$$\begin{array}{rcl}Hom_{(Polytope)}(S, T) \times Hom(T, X) \times Hom_{(k\text{-}Dg)}(X, Y) & \to & Hom(S, Y) \\ (a, \phi, \psi) & \to & \psi \circ \phi \circ a\end{array}$$

Note that these correspondences are compatible with the analytification. The underlying continuous map of a morphism to a dagger space coincides with that of the analytification, the completion of the bounded homomorphism between global sections associated a morphism to an affinoid dagger space is the bounded homomorphism between global sections associated with the analytification, and so on.

Proposition 2.4.12. *Let S_k and T_k be k-affinoid simplices, and $\mathscr{M}(A)$ and $\mathscr{M}(B)$ k-affinoid dagger spaces. The diagram*

$$\begin{array}{ccc}Hom_{(Polytope)}(S, T) \times Hom(T, \mathscr{M}(A)) \times Hom_{(k\text{-}Dg)}(\mathscr{M}(A), \mathscr{M}(B)) & \xrightarrow{\circ \times \circ} & Hom(S, \mathscr{M}(B)) \\ \Big\downarrow {\scriptstyle * \times H^0(-, \mathbb{G}_a^\dagger) \times H^0(-, \mathbb{G}_a^\dagger)} & & \Big\downarrow {\scriptstyle H^0(-, \mathbb{G}_a^\dagger)} \\ Hom_{(k\text{-}WBanach)}(k_T^\dagger, k_S^\dagger) \times Hom_{(k\text{-}WBanach)}(A, k_T^\dagger) \times Hom_{(k\text{-}WBanach)}(B, A) & \xrightarrow{\circ \times \circ} & Hom_{(k\text{-}WBanach)}(B, k_S)\end{array}$$

commutes.

Proposition 2.4.13. *The map*

$$\circ: \bigsqcup_{A \in \mathscr{A}_k^\dagger} Hom(S, \mathscr{M}(A)) \times Hom_{(k\text{-}Dg)}(\mathscr{M}(A), X) \to Hom(S, X)$$

induced by the composition

$$\begin{array}{rcl}\circ: Hom(S, \mathscr{M}(A)) \times Hom_{(k\text{-}Dg)}(\mathscr{M}(A), X) & \to & Hom(S, X) \\ (\gamma, \phi) & \mapsto & \phi \circ \gamma\end{array}$$

is surjective.

Definition 2.4.14. *For a morphism $\gamma \colon S \to X$, let $\Lambda(\gamma)$ be the collection of pairs (T, V) of a subpolytope $T \leq S$ and an analytic domain $V \subset X$ such that there exist a representative $\underline{\gamma} \in Hom(S, \mathscr{A}_k^{\dagger}, X)$ of γ and a ratinal domain $W \subset \mathscr{M}(A_{\underline{\gamma}})$ such that $\mathscr{M}(\underline{\gamma}^{(0)})^{\sharp}(|T_k|) \subset |W| \subset |\mathscr{M}(A_{\underline{\gamma}})|$ and $\underline{\gamma}^{(1)\sharp}(|W|) \subset |V| \subset |X|$.*

Proposition 2.4.15. *For a morphism $\gamma \colon S \to X$, there is a canonical functorial system $(\gamma^a_{T,V})_{(T,V)\in\Lambda(\gamma)}$ of k-algebra homomorphisms $\gamma^a_{T,V} \colon H^0(V, O_X^{\dagger}) \to k_T^{\dagger}$ satisfying the following properties:*

(i) for pairs $(T, V), (T', V') \in \Lambda(\gamma)$ satisfying $T' \leq T$ and $V \subset V'$, the diagram

$$\begin{tikzcd} k_T^{\dagger} \arrow[d] & H^0(V, O_X^{\dagger}) \arrow[l, "\gamma^a_{T,V}"'] \\ k_{T'}^{\dagger} & H^0(V', O_X^{\dagger}) \arrow[l, "\gamma^a_{T',V'}"'] \arrow[u] \end{tikzcd}$$

commutes.

(ii) for pairs $(T, V), (T', V') \in \Lambda(\gamma)$ satisfying $T' \leq T$ and $V' \subset V$, the diagram

$$\begin{tikzcd} k_T^{\dagger} \arrow[d] & H^0(V, O_X^{\dagger}) \arrow[l, "\gamma^a_{T,V}"'] \arrow[d] \\ k_{T'}^{\dagger} & H^0(V', O_X^{\dagger}) \arrow[l, "\gamma^a_{T',V'}"'] \end{tikzcd}$$

commutes.

(iii) for a pair $(T, V) \in \Lambda(\gamma)$, the diagram

$$\begin{tikzcd} |T_k| \arrow[r, "\mathscr{M}(\gamma^a_{T,V})^{\sharp}"] \arrow[d] & |V| \arrow[d] \\ |S_k| \arrow[r, "\gamma^{\sharp}"] & |X| \end{tikzcd}$$

commutes.

(iv) for a pair $(T, V) \in \Lambda(\gamma)$ satisfying that $V \subset X$ is a special domain, the k-algebra homomorphism $\gamma^a_{T,V} \colon A_V = H^0(V, O_X^{\dagger}) \to k_T^{\dagger}$ is bounded.

Proposition 2.4.16 (ground field extension)**.** *Let K/k be an extension of complete non-Archimedean fields, S_k a k-dagger simplex, and X a k-dagger space. The ground field extensions*

$$(\cdot)_K \colon Hom_{(k\text{-}WBanach)}(A, k_S^{\dagger}) \to Hom_{(K\text{-}WBanach)}(A_K, K_S^{\dagger})$$

$$\text{and} \qquad (\cdot)_K \colon \mathit{Hom}_{(k\text{-}Dg)}(\mathscr{M}(A), X) \to \mathit{Hom}_{(K\text{-}Dg)}(\mathscr{M}(A)_K, X_K)$$

for a k-dagger algebra $A \in \mathscr{A}_k^\dagger$ *induce the correspondence*

$$(\cdot)_K \colon \mathit{Hom}(S, X) \to \mathit{Hom}(S, X_K).$$

Also call it the ground field extention.

Proposition 2.4.17 (Galois action)**.** *The Galois action*

$$\begin{array}{rcl} \mathit{Hom}_{(k\text{-}WBanach)}(A, k_S^\dagger) \times G_k & \to & \mathit{Hom}_{(k\text{-}WBanach)}(A, k_S^\dagger) \\ (\phi, g) & \mapsto & g \circ \phi \end{array}$$

for a k-dagger algebra $A \in \mathscr{A}_k^\dagger$ *induce the Galois action*

$$\begin{array}{rcl} \mathit{Hom}(S, \mathscr{A}_k^\dagger, X) \times G_k & \to & \mathit{Hom}(S, \mathscr{A}_k^\dagger, X) \\ (\gamma, g) & \mapsto & \gamma \circ g \coloneqq (g^{-1} \circ \gamma^{(0)}, \gamma^{(1)}) \end{array}$$

which preserves the equivalence relation $\sim$*. Therefore it gives an well-defined Galois action*

$$\begin{array}{rcl} \mathit{Hom}(S, X) \times G_k & \to & \mathit{Hom}(S, X) \\ (\gamma, g) & \mapsto & \gamma \circ g. \end{array}$$

Proposition 2.4.18 (universality of the fibre product)**.** *Let* S_k *be a k-dagger simplex,* X, Y *k-dagger spaces, and* $\mathscr{M}(A)$ *a k-affinoid dagger space. For any morphisms* $\phi \colon X \to \mathscr{M}(A)$ *and* $\psi \colon Y \to \mathscr{M}(A)$*, one has the canonical functorial set-theoretical bijective map*

$$\mathit{Hom}(S, X) \times_{\mathit{Hom}(S, \mathscr{M}(A))} \mathit{Hom}(S, Y) \to \mathit{Hom}(S, X \times_{\mathscr{M}(A)} Y).$$

Proposition 2.4.19 (universality of the direct limit)**.** *Let* I *be a directed set,* S_k *a k-dagger simplex,* X *a k-dagger space, and* $W \colon I \to (k\text{-}Dg) \colon i \rightsquigarrow W_i$ *a direct system of k-analytic domains of* X *whose transitive map* $W_i \to W_j$ *is the inclusion* $W_i \hookrightarrow W_j \hookrightarrow X$ *for any* $i \leq j \in I$*. Suppose* W *converges to* X *and determines a topological covering of* X*. Then the canonical set-theoretical map*

$$\varinjlim_{i \in I} \mathit{Hom}(S, W_i) \to \mathit{Hom}(S, X)$$

induced by the composition $\mathit{Hom}(S, W_i) \to \mathit{Hom}(S, X)$ *of the embedding* $\iota_i \colon W_i \hookrightarrow X$ *is bijective.*

Corollary 2.4.20 (universality of the fibre product)**.** *Let* S_k *be a k-dagger simplex, and* X, Y, Z *k-dagger spaces. Suppose* Z *admits a direct system* $W \colon I \to (k\text{-}An) \colon i \rightsquigarrow W_i$ *of k-affinoid domains of* Z *satisfying the conditions that the transitive map* $W_i \to W_j$ *is the inclusion* $W_i \hookrightarrow W_j \hookrightarrow Z$ *for any* $i \leq j \in I$*, and that* W *converges to* Z *and determines a topological covering of* X*. For any morphisms* $\phi \colon X \to Z$ *and* $\psi \colon Y \to Z$*, one has the canonical functorial set-theoretical bijective map*

$$\mathit{Hom}(S, X) \times_{\mathit{Hom}(S, Z)} \mathit{Hom}(S, Y) \to \mathit{Hom}(S, X \times_Z Y).$$

Corollary 2.4.21. *Let S_k be a k-dagger simplex, $\{S_{1k}, \ldots, S_{mk}\} \in Cov_S(S_k)$, and X k-dagger space. Suppose X admits a direct system $W\colon I \to (k\text{-}Dg)\colon i \rightsquigarrow W_i$ of k-affinoid domains of X satisfying the conditions that the transitive map $W_i \to W_j$ is the inclusion $W_i \hookrightarrow W_j \hookrightarrow X$ for any $i \leq j \in I$, and that W converges to X and determines a topological covering of X. Then the sequence*

$$* \to Hom(S, X) \to \prod_{i=1}^{m} Hom(S_i, X) \rightrightarrows \prod_{i,j=1}^{m} Hom(S_i \wedge S_j, X)$$

is exact.

Proposition 2.4.22 (universality of a rational domain)**.** *Let S_k be a k-dagger simplex, A a k-dagger algebra, $V \subset \mathscr{M}(A)$ a rational domain, and $\gamma\colon S \to \mathscr{M}(A)$ a morphism. If the image $\gamma^\sharp(|S_k|) \subset |\mathscr{M}(A)|$ is contained in $|V| \subset |\mathscr{M}(A)|$, the morphism $\gamma\colon S \to \mathscr{M}(A)$ uniquely factors through the embedding $V \hookrightarrow \mathscr{M}(A)$.*

Proposition 2.4.23 (universality of a Weierstrass domain)**.** *Let S_k be a k-dagger simplex, A a k-dagger algebra, $V \subset \mathscr{M}(A)$ a Weierstrass domain, and $\gamma\colon S \to \mathscr{M}(A)$ a morphism. If the image $\gamma^\sharp(|S|) \subset |\mathscr{M}(A)|$ of the underlying polytope $S \subset S_k$ is contained in $|V| \subset |\mathscr{M}(A)|$, the morphism $\gamma\colon S \to \mathscr{M}(A)$ uniquely factors through the embedding $V \hookrightarrow \mathscr{M}(A)$.*

Definition 2.4.24. *Let X be a k-dagger space. An analytic domain $U \subset X$ is said to be a ratinal domain (or a Weierstrass domain) if for any morphism $\phi\colon \mathscr{M}(A) \to X$ from a k-affinoid space the pull-back $\phi^{-1}(U) \subset \mathscr{M}(A)$ is a ratinal domain (resp. a Weierstrass domain) of the affinoid dagger space $\mathscr{M}(A)$.*

Proposition 2.4.25 (universality of a rational domain)**.** *Let S_k be a k-dagger simplex, X a k-dagger space, $V \subset X$ a rational domain, and $\gamma\colon S \to X$ a morphism. If the image $\gamma^\sharp(|S_k|) \subset |X|$ is contained in $|V| \subset |X|$, the morphism $\gamma\colon S \to X$ uniquely factors through the embedding $V \hookrightarrow X$.*

Proposition 2.4.26 (universality of a Weierstrass domain)**.** *Let S_k be a k-dagger simplex, X a k-dagger space, $V \subset X$ a Weierstrass domain, and $\gamma\colon S \to X$ a morphism. If the image $\gamma^\sharp(|S|) \subset |X|$ of the underlying polytope $S \subset S_k$ is contained in $|V| \subset |X|$, the morphism $\gamma\colon S \to X$ uniquely factors through the embedding $V \hookrightarrow X$.*

Definition 2.4.27. *A k-dagger space X is said to be Stein if it admits an increasing filtration $X_0 \subset X_1 \subset \cdots \subset \cup X_i = X$ of X by affinoid domains X_i such that $X_i \subset Int(X_{i+1}/X)$ for each $i \in \mathbb{N}$ and X_i is a Weierstrass domain of X_{i+1} for each $i \in \mathbb{N}$, where $Int(X_{i+1}/X)$ is the relative interior. Call the sequence $X_0 \subset X_1 \subset \cdots \subset \cup X_i = X$ a Weierstrass filtration of X.*

Definition 2.4.28. *Let X be a k-dagger space, and $U \subset X$ an analytic domain. If U is Stein, call it a Stein subspace. Moreover a Stein subspace $U \subset X$ is said to be a Stein domain if U admits a Weierstrass filtration consisting of Weierstrass domains of X.*

Definition 2.4.29. *Let X be a k-dagger space. An analytic domain $U \subset X$ is said to be a great domain of X if U has the following universality. For any k-dagger simplex S_k and any morphism $\gamma \colon S \to X$, if the image $\gamma^\sharp(|S|) \subset |X|$ of the underlying polytope $S \subset S_k$ is contained in $|U| \subset |X|$, the morphism $\gamma \colon S \to X$ uniquely factors through the inclusion $U \hookrightarrow X$. If $U_K \subset X_K$ is a great domain for any extention K/k of complete non-Archimedean fields, then we say $U \subset X$ is a universally great domain.*

Proposition 2.4.30. *A Stein domain is a universal great domain.*

2.5 Overconvergent calculations

In this subsection, we calculate the set $\mathrm{Hom}(S, X)$ for significant examples of dagger spaces X in the totally same way as we did in §1.5. The most essencial case is the case k is a local field, and hence suppose $\dim_{\mathbb{Q}} \sqrt{|k^\times|} < \infty$ throughout this subsection.

Lemma 2.5.1. *The set-theoretical map*

$$\begin{array}{rcl} Hom_{(k\text{-}WBanach)}(k\{d^{-1}T\}^\dagger, k_S^\dagger) & \to & \left\{ f \in (k_S^\dagger)^m \,\middle|\, \|f\|_S \leq d \right\} \\ \phi & \mapsto & (\phi(T_1), \ldots, \phi(T_m)) \end{array}$$

induces the canonical bijective map

$$Hom(S, D_k^m(d)^\dagger) \to \left\{ f \in (k_S^\dagger)^m \,\middle|\, \|f\|_S \leq d \right\},$$

where $d = (d_1, \ldots, d_m) \in (0, \infty)^m$ and $D_k^m(d)^\dagger$ is the polydisc $\mathscr{M}(k\{d^{-1}T\}^\dagger)$.

Proof. We have only to show that the map

$$\begin{array}{rcl} \mathrm{Hom}_{k\text{-}\mathrm{WBanach}}(\mathrm{D}_k^1(d_1)^\dagger, k_S^\dagger) & \to & \left\{ f \in k_S^\dagger \,\middle|\, \|f\|_S \leq d_1 \right\} \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is bijective, because

$$k\{d^{-1}T\}^\dagger \cong_k k\{d_1^{-1}T_1\}^\dagger \otimes_k^\dagger \cdots \otimes_k^\dagger k\{d_m^{-1}T_m\}^\dagger.$$

It follows from the general fact of a bounded k-algebra homomorphism between weakly complete k-algebras. For a uniform weakly complete k-algebra A, we verify that the set-theoretical map

$$\begin{array}{rcl} \mathrm{Hom}_{(k\text{-}\mathrm{WBanach})}(k\{d_1^{-1}T_1\}^\dagger, A) & \to & A \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is injective and its image coincides with the subset $\{f \in A \mid \|f\| \leq d_1\} \subset A$. The map is injective because $\phi(T_1) \in A$ determines the restriction of ϕ on the dense k-subalgebra $k[T_1] \subset k\{d_1^{-1}T_1\}^\dagger$ for a bounded k-algebra homomorphism $\phi \colon k\{d_1^{-1}T_1\}^\dagger \to A$. For a bounded k-algebra homomorphism $\phi \colon k\{d_1^{-1}T_1\}^\dagger \to A$, and suppose $f := \phi(T_1) \in A$

satisfies $\|f\| > d_1$. Take a rational number $q \in \mathbb{Q}$ such that $\|f\| > |p|^q > d_1$, and present $q = a/b$ by integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}_+$. Set

$$F := \sum_{i=0}^{\infty} p^{-ai} T_1^{bi} \in k[[T_1]].$$

Since $\|f\|^b > |p|^a > d_1^b$, one has $F \in k\{d_1^{-1} T_1\}^\dagger$ and

$$\|p^{-ai} f^{bi}\| = (|p|^{-a} \|f\|^b)^i \xrightarrow{i \to \infty} \infty$$

by the uniformity of A. It follows that the infinite sum

$$\sum_{i=0}^{\infty} p^{-ai} f^{bi} = \sum_{i=0}^{\infty} p^{-ai} \phi(T_1)^{bi} = \lim_{n \to \infty} \phi \left(\sum_{i=0}^{n} p^{-ai} T_1^{bi} \right)$$

does not converge in A, and it contradicts the boundedness of ϕ. Therefore $\|\phi(T_1)\| \leq d_1$. We construct the inverse map $\{f \in A \mid \|f\| \leq d_1\} \to \mathrm{Hom}_{(k\text{-WBanach})}(k\{d_1^{-1} T_1\}^\dagger, A)$. Take an element $f \in A$ such that $\|f\| \leq d_1$, and consider the k-algebra homomorphism $\phi \colon k[T_1] \to A \colon T_1 \mapsto f$. The homomorphism ϕ has the unique bounded extension $\phi \colon k\{d_1^{-1} T_1\}^\dagger \hookrightarrow k\{\|f\|^{-1} T_1\}^\dagger \to A$ by the definition of the weak completeness, and hence one obtains a set-theoretical map $\{f \in A \mid \|f\| \leq d_1\} \to \mathrm{Hom}_{(k\text{-WBanach})}(k\{d_1^{-1} T_1\}^\dagger, A)$. Obviously this is the inverse map. $\square$

Corollary 2.5.2. *The bijective map above induces the canonical bijective maps*

$$Hom(S, \mathbb{A}_k^{m\dagger}) \cong (k_S^\dagger)^m$$

and

$$Hom(S, \mathring{D}_k^m(d)^\dagger) \cong \left\{ f \in (k_S^\dagger)^m \,\middle|\, \|f\|_S < d \right\},$$

where $\mathring{D}_k^m(d)^\dagger$ is the open disc $\cup_{d' < d} \mathscr{M}(k\{d'^{-1} T\}^\dagger)$ and $\mathbb{A}_k^{m\dagger}$ is the affine space $\cup \mathscr{M}(k\{d^{-1} T\}^\dagger)$.

Lemma 2.5.3. *The set-theoretical map*

$$\begin{array}{rcl} Hom_{(k\text{-}WBanach)}(k\{T_1, T_1^{-1}, \ldots, T_m, T_m^{-1}\}^\dagger, k_S^\dagger) & \to & ((k_S^\dagger)^{\circ\times})^m \\ \phi & \mapsto & (\phi(T_1), \ldots, \phi(T_m)) \end{array}$$

induces the canonical bijective map

$$Hom(S, A_k^m(1,1)^\dagger) \to ((k_S^\dagger)^{\circ\times})^m,$$

where $A_k^m(1,1)^\dagger$ is the torus $\mathscr{M}(k\{T_1, T_1^{-1}, \ldots, T_m, T_m^{-1}\}^\dagger)$.

Proof. It is reduced to the next lemma. $\square$

Lemma 2.5.4. *The set-theoretical map*

$$\begin{array}{rcl} Hom_{(k\text{-}Banach)}(k\{d^{(1)-1}T, d^{(-1)}T^{-1}\}^{\dagger}, k_S^{\dagger}) & \to & \left\{ f \in (k_S^{\dagger})^{\times m} \;\middle|\; \|f_i^{\sigma}\|_S \leq d_i^{(\sigma)}, {}^{\forall}i = 1, \ldots, m, {}^{\forall}\sigma = \pm 1 \right\} \\ \phi & \mapsto & (\phi(T_1), \ldots, \phi(T_m)) \end{array}$$

induce the canonical bijective maps

$$Hom(S, A_k^m(d^{(-1)}, d^{(1)})^{\dagger}) \to \left\{ f \in (k_S^{\dagger})^{\times m} \;\middle|\; \|f_i^{\sigma}\|_S \leq d_i^{(\sigma)}, {}^{\forall}i = 1, \ldots, m, {}^{\forall}\sigma = \pm 1 \right\}$$

and

$$Hom(S, \mathring{\mathbb{A}}_k(d^{(-1)}, d^{(1)})^{\dagger}) \to \left\{ f \in (k_S^{\dagger})^{\times m} \;\middle|\; \|f_i^{\sigma}\|_R < d_i^{(\sigma)}, {}^{\forall}i = 1, \ldots, m, {}^{\forall}\sigma = \pm 1 \right\},$$

where $A_k^m(d^{(-1)}, d^{(1)})^{\dagger}$ *and* $\mathring{\mathbb{A}}_k(d^{(-1)}, d^{(1)})^{\dagger}$ *are the annuli* $\mathscr{M}(k\{d_i^{(1)-1}T_i, d_i^{(-1)}T_i^{-1} \mid i = 1, \ldots, m\}^{\dagger})$ *and* $\varinjlim \mathscr{M}(k\{d_i'^{(1)-1}T_i, d_i'^{(-1)}T_i^{-1} \mid i = 1, \ldots, m\}^{\dagger})$ *respectively, and where* $d_i'^{(-1)}$ *and* $d_i'^{(1)-1}$ *in the limit run through all pair* $(d_i'^{(-1)}, d_i'^{(1)-1}) \in (0, \infty)^2$ *such that* $d_i^{(-1)} < d_i'^{(-1)} \leq d_i'^{(1)-1} < d_i^{(1)-1}$.

Proof. We have only to show that the map

$$\begin{array}{rcl} \mathrm{Hom}(S, \mathrm{A}_k^1(d_1^{(-1)}, d_1^{(1)})^{\dagger}) & \to & \left\{ f \in (k_S^{\dagger})^{\times} \;\middle|\; \|f_1^{\sigma}\|_S \leq d_1^{(\sigma)}, {}^{\forall}\sigma = \pm 1 \right\} \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is bijective, because

$$k\{d^{(1)-1}T, d^{(-1)}T^{-1}\}^{\dagger} \cong_k k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^{\dagger} \otimes_k^{\dagger} \cdots \otimes_k^{\dagger} k\{d_m^{(1)-1}T_m, d_m^{(-1)}T_m^{-1}\}^{\dagger}.$$

It follows from the general fact of a bounded k-homomorphism of weakly complete k-algebras. For a uniform wekly complete k-algebra A, we verify that the map

$$\begin{array}{rcl} \mathrm{Hom}_{(k\text{-WBanach})}(k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^{\dagger}, A) & \to & A \\ \phi & \mapsto & \phi(T_1) \end{array}$$

is injective and its image coincides with the subset $\{f \in A^{\times} \mid \|f^{\sigma}\| \leq d_1^{(\sigma)}, {}^{\forall}\sigma = \pm 1\} \subset A$. To begin with, the image $\phi(T_1) \in A$ is invertible because $T_1 \in (k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^{\dagger})^{\times}$ for a bounded k-algebra homomorphism $\phi \colon k\{T_1, T_i^{-1}\}^{\dagger} \to A$. The map is injective because $\phi(T_1) \in A^{\times}$ determines the restriction of ϕ on the dense k-subalgebra $k[T_1, T_1^{-1}] \subset k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^{\dagger}$ for a bounded k-algebra homomorphism $\phi \colon k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^{\dagger} \to A$. Take a bounded k-algebra homomorphism $\phi \colon k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^{\dagger} \to A$, set $f := \phi(T_1) \in A^{\times}$. Consider the composition

$$k\{d_1^{(1)-1}T_1\}^{\dagger} \hookrightarrow k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^{\dagger} \xrightarrow{\phi} A,$$

one obtains $\|f\| \leq d_1^{(1)}$ by Lemma 1.5.1. Similarly one has $\|f^{-1}\| \leq d_1^{(-1)}$, and the image of the map $\mathrm{Hom}(S, A_k^m(d^{(-1)}, d^{(1)})^{\dagger}) \to A \colon \phi \mapsto \phi(T_1)$ is contained in the subset $\{f \in A^{\times} \mid$

$\|f^\sigma\| \leq d_1^{(\sigma)}, {}^\forall\sigma = \pm 1\}$. We construct the inverse map $\{f \in A^\times \mid \|f^\sigma\| \leq d_1^{(\sigma)}, {}^\forall\sigma = \pm 1\} \to \mathrm{Hom}_{(k\text{-WBanach})}(k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^\dagger, A)$. Take an element $f \in A^\times$ such that $\|f\| \leq d_1^{(1)}$ and $\|f^{-1}\| \leq d_1^{(-1)}$, and consider the k-algebra homomorphism

$$\begin{array}{rcl} \phi\colon k[T_1, T_2] & \to & A \\ T_1 & \mapsto & f \\ T_2 & \mapsto & f^{-1} \end{array}$$

The homomorphism ϕ has the unique bounded extension $\phi\colon k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_2\}^\dagger \hookrightarrow k\{\|f\|^{-1}T_1, \|f^{-1}\|^{-1}T_2\}^\dagger \to A$ by the definition of the weak completeness, and its kernel contains $T_1T_2 - 1$. Therefore it uniquely factors through the quotient

$$k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_2\} \to k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_2\}/(T_1T_2 - 1) = k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\},$$

and one obtains a set-theoretical map

$$\left\{ f \in A^\times \,\middle|\, \|f^\sigma\| \leq d_1^{(\sigma)}, {}^\forall\sigma = \pm 1 \right\} \to \mathrm{Hom}_{(k\text{-WBanach})}(k\{d_1^{(1)-1}T_1, d_1^{(-1)}T_1^{-1}\}^\dagger, A).$$

Obviously this is the inverse map. □

Corollary 2.5.5. *The bijective maps above induce the canonical bijective map*

$$\mathit{Hom}(S, \mathbb{G}_{m,k}^\dagger) \cong (k_S^\dagger)^\times,$$

where $\mathbb{G}_{m,k}^\dagger$ is the open torus $\cup\mathscr{M}(k\{d^{-1}T_1, d'T_1^{-1}\}^\dagger)$.

In order to determines the multiplicative groups $(k_S^\dagger)^{\circ\times}$ and $(k_S^\dagger)^\times$ using the result of the calculations of $(k_S)^{\circ\times}$ and $(k_S)^\times$, we verify the commutativity of the operations $\dagger, \circ, \circ\times, \times$.

Lemma 2.5.6.

$$\begin{array}{rcl} (k_S)^\circ \cap k_S^\dagger & = & (k_S)^\circ \\ (k_S)^{\circ\circ} \cap k_S^\dagger & = & (k_S)^{\circ\circ} \\ (k_S)^{\circ\times} \cap k_S^\dagger & = & (k_S^\dagger)^{\circ\times} \\ (k_S)^\times \cap k_S^\dagger & = & (k_S^\dagger)^\times. \end{array}$$

Proof. We may and do assume that S is thick. The first and the second assertions are trivial, and the third one follows from the fourth one. The inclusion $(k_S^\dagger)^\times \subset (k_S)^\times \cap k_S^\dagger$ is trivial. Take an element $f \in (k_S)^\times \cap k_S^\dagger$, and it suffice to show that $f \in (k_S^\dagger)^\times$, or equvalently $f^{-1} \in k_S^\dagger$. By the structure theorem of $k_S^\times$, Proposition 1.5.13, one has

$$E_{k,n} \times (k^\times + (k_S)^{\circ\circ}) = (k_S)^\times,$$

and hence there are unique elements $x \in E_{k,n}$, $u \in k^\times$, and $g \in (k_S)^{\circ\circ}$ such that $f = x(u+g)$. Since $E_{k,n}, k^\times \subset k_S^\dagger$, it follows that $g \in (k_S^\dagger)^{\circ\circ}$, and hence $f^{-1} = x^{-1}u^{-1}(1 + u^{-1}g)^{-1} \in k_S^\dagger$ by the following lemma. □

Lemma 2.5.7. *For an weakly complete k-algebra A, one has the inclusion*

$$1 + A^{\circ\circ} \subset A^{\times}.$$

Note that it is well-known that the same inclusion holds for a Banach algebra.

Proof. Take an element $f \in 1 + A^{\circ\circ}$. Since $\|f - 1\| < 1$, the formal power series

$$F := \sum_{i=0}^{\infty} T_1^i \in k[[T_1]]$$

is an overconvergent power series on the closed disc of radius $\|f - 1\|$. Therefore the infinite sum

$$F(f - 1) = \sum_{i=0}^{\infty} (f - 1)^i$$

converges in A by the weak completeness of A, and it coincides with $f^{-1} \in \hat{A}$ by the well-known fact of a k-Banach algebra. Therefore $f^{-1} \in A$. □

Corollary 2.5.8. *Let $S \subset \mathbb{R}^n$ be a thick polytope containing $(0, \ldots, 0) \in \mathbb{R}^n$. An element $f \in (k_S^{\dagger})^{\circ}$ is invertible in $(k_S^{\dagger})^{\circ}$ if and only if there uniquely exists an element $x \in E_{k,n}$ such that $|f_x| = 1, \|x\|_S = \|x^{-1}\|_S = 1$, and $\|f_x^{-1} x^{-1} f - 1\|_S < 1$. In other words, the canonical homorphism*

$$\begin{array}{rcl} k^{\circ\times} \times (E_1)^n \times (1 + (k_S^{\dagger})^{\circ\circ}) & \to & (k_S^{\dagger})^{\circ\times} \\ (a, x, 1 + g) & \mapsto & ax(1 + g) \end{array}$$

is an isomorphism.

Proof. Trivial because $k^{\circ\times}, E_{k,n} \subset k_S^{\dagger}$, $(k_S)^{\circ} \cap k_S^{\dagger} = (k_S^{\dagger})^{\circ}$ and $(k_S)^{\circ\times} \cap k_S^{\dagger} = (k_S^{\dagger})^{\circ\times}$. □

Corollary 2.5.9. *The set-theoretical bijective map*

$$\begin{array}{rcl} Hom_{(k\text{-}WBanach)}(k\{T_1, T_1^{-1}\}^{\dagger}, k_S^{\dagger}) & \to & (k_S^{\dagger})^{\circ\times} \\ \phi & \mapsto & \phi(T_1) \end{array}$$

and the canonical isomorphism

$$\begin{array}{rcl} k^{\circ\times} \times (E_1)^n \times (1 + (k_S^{\dagger})^{\circ\circ}) & \to & (k_S^{\dagger})^{\circ\times} \\ (a, x, 1 + g) & \mapsto & ax(1 + g) \end{array}$$

induce the canonical bijective map

$$Hom(S, A_k^1(1,1)^{\dagger}) \to k^{\circ\times} \times (E_1)^n \times (1 + (k_S^{\dagger})^{\circ\circ}).$$

Corollary 2.5.10. *Let $S \subset \mathbb{R}^n$ be a thick polytope containing* $(0, \ldots, 0) \in \mathbb{R}^n$. *An element $f \in k_S^\dagger$ is invertible if and only if there uniquely exists an $x \in E_{k,n}$ such that $|f_x| \|x\|_S = \|f\|_S$ and $\|f_x^{-1} x^{-1} f - 1\|_S < 1$. In other words, the canonical homorphism*

$$\begin{array}{rcl} k^\times \times E_{k,n} \times (1 + (k_S^\dagger)^{\circ\circ}) & \to & (k_S^\dagger)^\times \\ (a, x, 1 + g) & \mapsto & ax(1 + g) \end{array}$$

is an isomorphism.

Proof. Trivial because $k^\times, E_{k,n} \subset k_S^\dagger$ and $(k_S)^{\circ\circ} \cap k_S^\dagger = (k_S^\dagger)^{\circ\circ}$ and $(k_S)^\times \cap k_S^\dagger = (k_S^\dagger)^\times$. $\square$

Corollary 2.5.11. *The bijective maps above induce the canonical bijective map*

$$Hom(S, \mathbb{G}_{m,k}^\dagger) \cong (k_S^\dagger)^\times = k^\times \times E_{k,n} \times (1 + (k_S^\dagger)^{\circ\circ}).$$

Moreover if the valuation of k is discrete and k is perfect, then one has

$$[\tilde{k}^\times] \times \pi_k^{\mathbb{Z}} \times E_{k,n} \times (1 + (k_S^\dagger)^{\circ\circ}) \cong Hom(S, \mathbb{G}_{m,k}^\dagger),$$

where $\pi_k \in k$ is a uniformiser.

We finish the examples.

3 Singular homology

In this section, we introduce the notion of $2 \times 2 \times 2 = 8$ kinds of the singular homologies. The eight types are distinguished by whether an object is an analytic space or a dagger space, whether singular simplices are cubical or not, and whether geometric points are reflected or not. We call them generically the analytic homologies. Remark that the homologies of a dagger space reflect the overconvergent structure of it. Therefore it has relation with the theory of integration of an overconvergent analytic function. The integration will help us to calculate the cubical singular homology of a k-dagger space, because Stokes' theorem often tells us whether a given cycle in the homology is boundary or not. In this section, we do not make use of the integration, and hence many basic properties of the eight homologies will be simultaneously verified.

3.1 Eight singular homologies

In order to deal with an analytic space and a dagger space at the same time, we introduce a unified notation for them.

Definition 3.1.1. *The term "$\mathscr{A}$" implies "analytic" or "dagger". For example if $\mathscr{A}$ = analytic, the term "a k-$\mathscr{A}$ space" means a k-analytic space. Denote by k-$\mathscr{A}$ the category of k-$\mathscr{A}$ spaces.*

Definition 3.1.2. *If the residue field $\tilde{k}$ is a finite field, and set $q_k := \#\tilde{k} \geq 2 \in \mathbb{N}$ and $N_k := q_k - 1 \in \mathbb{N}_+$ or $N_k := 1 \in \mathbb{N}_+$. If the residue field $\tilde{k}$ is an infinite field, set $N_k := 1 \in \mathbb{N}_+$.*

In particular if k is a local field, we mainly assume $N_k = q_k - 1$.

Definition 3.1.3 (analytic singular homology)**.** *Let X a k-$\mathscr{A}$ space. Let $N_k\Delta^n \subset \mathbb{R}^{n+1}$ be the n-th normalised simplex*

$$N_k\Delta^n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, t_0 + \cdots + t_n = N_k, t_i \geq 0, {}^\forall i = 0, \ldots, n \right\}.$$

We call a homomorphism $N_k\Delta^n \to X$ a singular simplex of dimension n. Denote by $C_n^\triangle(X)$ the free $\mathbb{Z}$-module generated by the set of singular simplices of dimension n. For each analytic path $f\colon N_k\Delta^n \to X$, denote by $[f]$ the image of f by the set-theoretical maps

$$[\cdot]\colon Hom(N_k\Delta^n, X) \hookrightarrow C_n^\triangle(X).$$

Define the integral affine maps (Definition 1.1.16) $\partial_n^{(i)}\colon N_k\Delta^{n-1} \to N_k\Delta^n$ for each $i = 0, \ldots, n$ by

$$\begin{array}{rcl} N_k\Delta^{n-1} & \to & N_k\Delta^n \\ (t_0, \ldots, t_{n-1}) & \mapsto & (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, s_{n-1}). \end{array}$$

We also have an integral affine map

$$\begin{array}{rcl} \sigma_n^{(i)}\colon N_k\Delta^n & \to & N_k\Delta^{n-1} \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_n) \end{array}$$

for each $i = 0, \ldots, n-1$. Then for each $i = 0, \ldots, n-1$ and $j = 0, \ldots, n$, one has

$$\sigma_n^{(i)} \circ \partial_n^{(j)} = \begin{cases} \partial_{n-1}^{(j)} \circ \sigma_{n-1}^{(i-1)} & (\text{if } 0 \leq j \leq i-1) \\ id & (\text{if } j = i, i+1) \\ \partial_{n-1}^{(j-1)} \circ \sigma_{n-1}^{(i)} & (\text{if } i+1 \leq j \leq n). \end{cases}$$

Define the homomorphism $d_n\colon C_n^\triangle(X) \to C_{n-1}^\triangle(X)$ by

$$\gamma \mapsto \sum_{i=0}^{n} (-1)^i \gamma \circ \partial_n^{(i)}$$

for each $\gamma\colon N_k\Delta^n \to X$. The homomorphism d_n gives us the chain complex $C_\cdot^\triangle(X)$. Denote by $C_n^\triangle(X, M)$ the $\mathbb{Z}$-module $C_n^\triangle(X) \otimes_{\mathbb{Z}} M$ for an Abelian group M, and call it the group of singular chains of dimension n with coefficients in M. Particularly $C_n^\triangle(X, \mathbb{Z}) = C_n^\triangle(X)$. We set $H_n^\triangle(X, M) := H_n^\triangle(C_\cdot^\triangle(X, M))$ and call it the n-th analytic singular homology group of X with coefficients in M. On the other hand, define the group of singular cochains of dimension n with coefficients in M as $C_\triangle^n(X, M) := Hom_{Grp}(C_n^\triangle(X), M)$. Call $H_\triangle^n(X, M) := H^n(C_\triangle^\cdot(X, M))$ the n-th analytic singular cohomology group of X with coefficients in M. If the coefficient group M is $\mathbb{Z}$, we write $H_n^\triangle(X)$ and $H_\triangle^n(X)$ instead of $H_n^\triangle(X, \mathbb{Z})$ and $H_\triangle^n(X, \mathbb{Z})$ for short.

Definition 3.1.4 (analytic cubical singular homology)**.** *Let X a k-$\mathscr{A}$ space. Following Serre's convention in [SER1], we call a morphism $[0, N_k]^n \to X$ a singular cube of dimension n. A singular cube $\gamma\colon [0, N_k]^n \to X$ of dimension n is said to be degenerate if there exist an integer $i = 1, \ldots, n$ and a singular cube $\gamma'\colon [0, N_k]^{n-1} \to X$ of dimension $n-1$ such that γ coincides with $\gamma' \circ pr_i^{(n)}$, where $pr_i^{(n)}$ is the integral affine maps $[0, N_k]^n \to [0, N_k]^{n-1}$ determined by the continuous map endowsed with presentation by the natural projection matrix below:*

$$\begin{array}{rcl} [0, N_k]^n & \to & [0, N_k]^{n-1} \\ t = (t_1, \ldots, t_n) & \mapsto & (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n). \end{array}$$

Denote by $Q_n^{\square}(X)$ the free $\mathbb{Z}$-module generated by the set of singular cubes of dimension n, and by $D_n^{\square}(X) \subset Q_n^{\square}(X)$ the $\mathbb{Z}$-submodule generated by degenerate singular cubes. The group $C_n^{\square}(X) := Q_n^{\square}(X)/D_n^{\square}(X)$ of singular cubical chains of X is isomorphic to the free $\mathbb{Z}$-module generated by the set of non-degenerate singular cubes. For each analytic path $f\colon [0, N_k]^n \to X$, denote by $[f]$ the image of f by the set-theoretical maps

$$[\cdot]\colon Hom([0, N_k]^n, X) \hookrightarrow Q_n^{\square}(X)$$

and

$$[\cdot]\colon Hom([0, N_k]^n, X) \hookrightarrow Q_n^{\square}(X) \twoheadrightarrow C_n^{\square}(X).$$

Define the integral affine maps $\partial_n^{(i,\sigma)}\colon [0, N_k]^{n-1} \to [0, N_k]^n$ for each $i = 1, \ldots, n$ and $\sigma = 0, 1$ by

$$\begin{array}{rcl} [0, N_k]^{n-1} & \to & [0, N_k]^n \\ (t_1, \ldots, t_{n-1}) & \mapsto & (t_1, \ldots, t_{i-1}, N_k\sigma, t_i, \ldots, t_{n-1}). \end{array}$$

Define the homomorphism $d_n\colon Q_n^{\square}(X) \to Q_{n-1}^{\square}(X)$ by

$$\gamma \mapsto \sum_{i=1}^{n} \sum_{\sigma=0}^{1} (-1)^{i+\sigma} \gamma \circ \partial_n^{(i,\sigma)}$$

for each $\gamma\colon [0, N_k]_k^n \to X$. The induced homomorphism $d_n\colon C_n^{\square}(X) \to C_{n-1}^{\square}(X)$ gives us the chain complex $C_{\cdot}^{\square}(X)$. Denote by $C_n^{\square}(X, M)$ the $\mathbb{Z}$-module $C_n^{\square}(X) \otimes_{\mathbb{Z}} M$ for any Abelian group M, and call it the group of singular cubical chains of dimension n with coefficients in M. Particularly $C_n^{\square}(X, \mathbb{Z}) = C_n^{\square}(X)$. We set $H_n^{\square}(X, M) := H_n^{\square}(C_{\cdot}^{\square}(X, M))$ and call it the n-th analytic cubical singular homology group of X with coefficients in M. On the other hand, define the group of singular cubical cochains of dimension n with coefficients in M as $C_{\square}^n(X, M) := Hom_{Grp}(C_n^{\square}(X), M)$. Call $H_{\square}^n(X, M) := H^n(C_{\square}^{\cdot}(X, M))$ the n-th analytic cubical singular cohomology group of X with coefficients in M. If the coefficient group M is $\mathbb{Z}$, we write $H_n^{\square}(X)$ and $H_{\square}^n(X)$ instead of $H_n^{\square}(X, \mathbb{Z})$ and $H_{\square}^n(X, \mathbb{Z})$ for short.

The normalisation by the constant N_k is quite essential when we consider the integration of a differential form along a cycle in the analytic (cubical) singular homology. Recall we verified the integrability of an overconvergence analytic function on $[0, q_k - 1]$, but not on $[0, 1]$. The end point must be $q_k - 1 \in \mathbb{R}$ so that Stokes' theorem holds. If we set the end point of an interval as $0, 1 \in \mathbb{R}$, we would lack Stoke's theorem. See the proof of the fundamental theorem of calculus, Proposition 4.2.32.

Definition 3.1.5. *The symbol "C" (or "H") implies "$C^{\triangle}$" or "$C^{\square}$" (resp. "$H^{\triangle}$" or "$H^{\square}$"). For example if $H = H^{\triangle}$, the symbol "$H_n(X, M)$" means "$H_n^{\triangle}(X, M)$".*

Recall that the étale cohomology with coefficients in $\mathbb{Q}_l$ for a prime number $l \in \mathbb{N}$ is defined as the cohomology of a $\mathbb{Q}_l$-sheaf. This idea also works for the analytic homologies, and hence only when the coefficient group M is a $\mathbb{Q}_l$-Banach space B, we rearrange the definition of the analytic homologies with coefficients in B ragarding B as the tensor of $\mathbb{Q}_l$ and the pro-object $B^{\circ} = \varprojlim_{m \to \infty} B^{\circ}/l^m B^{\circ}$.

Definition 3.1.6. *For groups M, N and a group homomorphism $M \to N$, its tensor product $C_{\cdot}(X, M) \to C_{\cdot}(X, N)$ is obviously a homomorphism of chain complices, and hence associate it with the homomorphism*

$$H_*(X, M) \to H_*(X, N)$$

of homological functors.

Definition 3.1.7. *For a prime number $l \in \mathbb{N}$ and a $\mathbb{Q}_l$-Banach space B, set*

$$H^*(X, B) := \varprojlim_{m \to \infty} H^*(X, B^{\circ}/l^m B^{\circ}) \times_{\mathbb{Z}_l} \mathbb{Q}_l.$$

In particular set

$$H^*(X, \mathbb{Q}_l) := \varprojlim_{m \to \infty} H^*(X, \mathbb{Z}_l/l^m \mathbb{Z}_l) \times_{\mathbb{Z}_l} \mathbb{Q}_l = \varprojlim_{m \to \infty} H^*(X, \mathbb{Z}/l^m \mathbb{Z}) \times_{\mathbb{Z}_l} \mathbb{Q}_l.$$

We mainly make use of the analytic singular homology of a dagger space, and we do not know whether the analytic singular homology and the analytic cubical singular homology coincide. Maybe not. The canonical homomorphism from the analytic singular homology to the analytic cubical singular homology might not be surjective or injective. The analytic singular homology seems to have much more cycles and boundaries than the analytic cubical singular homology. Remark tha the existence of Mayer-Vietoris exact sequence will be easily proved for the analytic singular homology by the similar way as that of the corresponding fact in general topology. On the other hand, when we try to prove the existence of Mayer-Vietoris exact sequence for the analytic cubical singular homology, a homotopy between the identity and the barycentric subdivision operator can not be constructed as a cubical singular cube, and hence we have to extend the definition of a cubical singular cube, which will be called a "general" analytic cubical singular

cube. Moreover, the canonical homotopical isomorphism between cubical one to not cubical one can be obtained as a "general" cycle. Therefore the existence of Mayer-Vietoris exact sequence and the comparison isomorphism to not cubical one will be verified only for the "generalised" analytic cubical singular homology, but not for the analytic cubical singular homology. See Appendix, §6.1. for the theory of the "generalised" analytic homology. The reason why we mainly deal with the homologies of dagger spaces in this paper is because the integration is valid for them. The "generalised" ones do not work in the theory of integration by the following reason: Recall that we are regarding a character $x \in \mathbb{Q}_k^\vee$ as an analytic function on $[0, N_k]$. The "generalised" analytic (cubical) singular homology is obtained regarding a formal symbol $x(t^n)$ for a character $x \in \mathbb{Q}_k^\vee$ and an integer $n \in \mathbb{N}_+$ as a "general" analytic function on $[0, N_k]$. A primitive function of an exponential function a^{t_1} is an elementary function $\log a a^{t_1}$, but that of $a^{t_1^2}$ is not. Fubini's theorem help you to calculate the integral of $a^{t_1} b^{t_2}$ on $[0, N_k]^2$, but not of $a^{t_1 t_2}$. Singular cubes correspond to fuctions such as a^{t_1} and $a^{t_1} b^{t_2}$, while general singular cubes correspond to functions such as $a^{t_1^2}$ and $a^{t_1 t_2}$. See Appendix, §6.1. for more precise information.

To begin with, consider the case $N_k = 1$.

Definition 3.1.8 (ground field extension of analytic singular homology)**.** *Let K/k be an extension of complete non-Archimedean fields. The ground field extension functors*

$$\begin{array}{rcl} X & \rightsquigarrow & X_K \\ S_k & \rightsquigarrow & S_K \\ Hom(S, X) & \rightsquigarrow & Hom(S, X) \end{array}$$

induce canonical homomorphisms

$$\begin{array}{rcl} (\cdot)_K \colon C^{\triangle}_{\cdot}(X, M) & \to & C^{\triangle}_{\cdot}(X_K, M) \\ [\gamma] & \to & [\gamma_K] \\ \textit{and} \ \ (\cdot)_K \colon H^{\triangle}_{*}(X, M) & \to & H^{\triangle}_{*}(X_K, M). \end{array}$$

Definition 3.1.9 (ground field extension of analytic cubical singular homology)**.** *Let K/k be an extension of complete non-Archimedean fields. The ground field extension functors*

$$\begin{array}{rcl} X & \rightsquigarrow & X_K \\ S_k & \rightsquigarrow & S_K \\ Hom(S, X) & \rightsquigarrow & Hom(S, X) \end{array}$$

induce canonical homomorphisms

$$\begin{array}{rcl} (\cdot)_K \colon C^{\square}_{\cdot}(X, M) & \to & C^{\square}_{\cdot}(X_K, M) \\ [\gamma] & \to & [\gamma_K] \\ \textit{and} \ \ (\cdot)_K \colon H^{\square}_{\cdot}(X, M) & \to & H^{\square}_{\cdot}(X_K, M). \end{array}$$

We often need the assumption that k is algebraically closed to verify some important properties of the analytic (cubical) singular homology. The assumption is too strong because we mainly want to see the case k is a local field. If k is not algebraically closed, the analytic cubical singular homology $\mathrm{H}_*(X_C, M)$ may somtimes be easier to deal with, where C is the completion of the fixed algebraic closure $\overline{k}$ of k.Note that the biggest reason why we have to deal with a local field is because we will use Fontain's p-adic period ring B_{dR} in the theory of the integration, and the proof of the integrability of a differential form on a k-dagger space along a cycle in the sense of the analytic (cubical) singular homology heavily depends on the finiteness of the residue field $\tilde{k}$ and the discreteness of the valuation of k. Therefore when we consider the analytic cubical homology of a dagger space, we should not extend the base field to C. In the case, the following alternative notion may be helpful:

Definition 3.1.10 (geometric analytic homology)**.** *Let X be a k-$\mathscr{A}$ space. Set*

$$H_n(X/\overline{k}, M) \coloneqq \varinjlim_{\overline{k}/K/k} H_n(X_K, M),$$

where K in the limit runs through all finite extension K/k contained in the fixed algebraic closure $\overline{k}$. Remark that if k is algebraically closed, one has the canonical isomorphism $H_*(X, M) \cong H_*(X/\overline{k}, M)$.

Now consider the case k is a local field and $N_k = q_k - 1$.

Lemma 3.1.11. *Let K/k be an extension of local fields. Then one has $N_k \mid N_K$. If K contains a N_K/N_k-th root of a uniformiser of k, then one has $k(\sqrt[N_K/N_k]{k}) \subset K$.*

Proof. Trivial by the structure of the multiplicative group $k^\times = [\tilde{k}^\times] \times \pi_k^{\mathbb{Z}} \times (1 + k^{\circ\circ})$, where $\pi_k \in k$ is a uniformiser. Note that $N_K/N_k \equiv -1 \bmod . p$. □

Definition 3.1.12 (ground field extension of analytic singular homology)**.** *Let K/k be an extension of local fields. Suppose K contains a N_K/N_k-th root of a uniformiser of k. The ground field extension functors*

$$\begin{array}{rcl} X & \rightsquigarrow & X_K \\ S_k & \rightsquigarrow & S_K \\ Hom(S, X) & \rightsquigarrow & Hom(S, X) \end{array}$$

and the (not integral) affine maps

$$\begin{array}{rcl} N_k/N_K \colon N_K\Delta^n & \to & N_k\Delta^n \\ (t_0, \ldots, t_n) & \mapsto & \dfrac{N_k}{N_K}(t_0, \ldots, t_n) \end{array}$$

induce canonical homomorphisms

$$(\cdot)_K \colon C^{\triangle}_{\cdot}(X, M) \to C^{\triangle}_{\cdot}(X_K, M)$$

$$\begin{array}{rcl} [\gamma] & \to & [\gamma_K] \\ and\ \ (\cdot)_K \colon H^{\triangle}_{*}(X, M) & \to & H^{\triangle}_{*}(X_K, M). \end{array}$$

Definition 3.1.13 (ground field extension of analytic cubical singular homology)**.** *Let K/k be an extension of local fields. Suppose K contains a N_K/N_k-th root of a uniformiser of k. The ground field extension functors*

$$\begin{array}{rcl} X & \rightsquigarrow & X_K \\ S_k & \rightsquigarrow & S_K \\ Hom(S, X) & \rightsquigarrow & Hom(S, X) \end{array}$$

and the (not integral) affine maps

$$\begin{array}{rcl} N_k/N_K \colon [0, N_K]^n & \to & [0, N_k]^n \\ (t_0, \ldots, t_n) & \mapsto & \dfrac{N_k}{N_K}(t_0, \ldots, t_n) \end{array}$$

induce canonical homomorphisms

$$\begin{array}{rcl} (\cdot)_K \colon C^{\square}_{\cdot}(X, M) & \to & C^{\square}_{\cdot}(X_K, M) \\ [\gamma] & \to & [\gamma_K] \\ and\ \ (\cdot)_K \colon H^{\square}_{*}(X, M) & \to & H^{\square}_{*}(X_K, M). \end{array}$$

As a redear sees, the ground field extension is bother some in the case $N_k = q_k - 1$. If one needs the ground field extension, he or she may change the homology in the following way:

Definition 3.1.14. *The integral affine maps*

$$\begin{array}{rcl} \Delta^n & \to & (q_k - 1)\Delta^n \\ (t_0, \ldots, t_n) & \mapsto & (q_k - 1)(t_0, \ldots, t_n) \end{array}$$

and

$$\begin{array}{rcl} [0, 1]^n & \to & [0, q_k - 1]^n \\ (t_1, \ldots, t_n) & \mapsto & (q_k - 1)(t_1, \ldots, t_n) \end{array}$$

induce canonical homomorphisms from the analytic (cubical) singular homology given by setting $N_k = q_k - 1$ to the analytic (cubical) singular homology given by setting $N_k = 1$.

We have constructed the eight singular homologies: the geometric and not geometric, cubical and not cubical, singular homologies of an analytic space and a dagger space. We call them generically the analytic homologies distinguishing the other homologies such as the singular homology of the underlying topological space. Finally we introduce the notion of the analytic homology of an algebraic variety.

Definition 3.1.15 (analytic homology)**.** *An analytic homology implies one of the eight homologies defined above.*

Definition 3.1.16 (analytic homology of an algebraic variety)**.** *For an algebraic variety X over k and an Abelian group M, denote by $H_*^{an}(X, M)$ (or $H_*^{\dagger}(X, M)$) the analytic (cubical) singular homology group $H_*(X^{an}, M)$ (resp. $H_*(X^{\dagger}, M)$) of the associated k-analytic space X^{an} (resp. the associated k-dagger space $X^{\dagger}$).*

3.2 Axiom of homology and other properties

The analytic homologies satisfies many desired properties containing what is called the axiom of homology. We verify the functoriality, universal coefficient theorem, existence of the long exact sequence for a space pair, dimension axiom, homotopy equivariance, existence of Mayer-Vietoris exaxt sequence, excision axiom, the relation between H_0 and some kinds of connectedness, and compatibility of the two group structure of H_1 of a group object. Be careful that the existence of Mayer-Vietoris exact sequence and the excision axiom hold only for the "generalised" analytic cubical singular homology of an $\mathscr{A}$ space over an algebraically closed field. We will explain why such a revision is necessary in Appendix, §6.1. In the proof of the propositions of the analytic homology, we only consider the case $N_k = 1$. Even if $N_k > 1$, the proof is completely the same.

Lemma 3.2.1 (functoriality)**.** *The correspondence*

$$H_n(\cdot, M) \colon (k\text{-}\mathscr{A}) \ \ \to \ \ (\mathbb{Z}\text{-}Mods)$$

is a covariant functor.

Proof. The correspondence of Hom-sets is given by the composition of a morphism in (k-$\mathscr{A}$) to a singular simplex (or a singular cube) in a natural way as we have dealt with in Proposition 1.4.11. and Proposition 2.4.11, and the functoriality holds. □

Proposition 3.2.2 (universal coefficient theorem)**.** *Let X be a k-$\mathscr{A}$ space. One has a splitting exact sequence*

$$0 \to H_n(X) \otimes_{\mathbb{Z}} M \to H_n(X, M) \to Tor^{\mathbb{Z}}(H_{n-1}(X), M) \to 0$$

for an integer $n \in \mathbb{N}$ and an Abelian group M.

Proof. It follows from the fact that $\mathrm{C}_n(X)$ is isomorphic to the free $\mathbb{Z}$-module generated by the set of non-degenerate singular cubes. □

Since we has the universal coefficient theorem, we argue only about the $\mathbb{Z}$-coefficient singular homology $\mathrm{H}_*(X)$. Now we will see several basic properties required for a homology theory in the next subsection.

Proposition 3.2.3 (long exact sequence for a space pair)**.** *Let X be a k-$\mathscr{A}$ space, and $A \subset X$ an analytic domain. Set $C_n(X,A) \coloneqq C_n(X)/C_n(A)$ and $H_n(X,A) \coloneqq H_n(C_*(X,A))$. Then there exists a long exact sequence*

$$H_{n+1}(X,A) \to H_n(A) \to H_n(X) \to H_n(X,A).$$

Proof. Since the sequence

$$0 \to \mathrm{C}_*(A) \to \mathrm{C}_*(X) \to \mathrm{C}_*(X,A) \to 0$$

is an exact sequence of shain complices of $\mathbb{Z}$-modules by the definition of $\mathrm{C}_*(X,A)$, it induces the long exact sequence taking its homology. □

Proposition 3.2.4 (dimension axiom)**.** *One has*

$$H_n(\mathscr{M}(k)) = \begin{cases} \mathbb{Z} & (n=0) \\ 0 & (n>0) \end{cases}$$

Note that k is both of a k-affinoid algebra and a k-dagger algebra, and the assertion holds for both of the analytic homologies of an analytic space and of a dagger space.

Proof. We omit the dagger case. Calculate the chain complex $\mathrm{C}_*(\mathscr{M}(k))$ in the following way: Take an arbitrary $n \in \mathbb{N}$. Since $\mathscr{M}(k)$ is a k-affinoid space, one has

$$\begin{aligned} \mathrm{Hom}([0,1]^n, \mathscr{M}(k)) &= \mathrm{Hom}_{(k\text{-NAlg})}(k, k_{[0,1]^n}) = \{\mathrm{id}_k\} \\ \mathrm{Hom}(\Delta^n, \mathscr{M}(k)) &= \mathrm{Hom}_{(k\text{-NAlg})}(k, k_{\Delta^n}) = \{\mathrm{id}_k\} \end{aligned}$$

by Proposition 2.4.10. It follows that the maps

$$\begin{array}{rrcl} \circ\partial_n^{(i,\sigma)} \colon & \mathrm{Hom}([0,1]^n, \mathscr{M}(k)) & \to & \mathrm{Hom}([0,1]^{n+1}, \mathscr{M}(k)) \\ & \gamma & \mapsto & \gamma \circ \partial_n^{(i,\sigma)} \\ \text{and} \quad \circ\, \partial_n^{(i)} \colon & \mathrm{Hom}(\Delta^n, \mathscr{M}(k)) & \to & \mathrm{Hom}(\Delta^{n+1}, \mathscr{M}(k)) \\ & \gamma & \mapsto & \gamma \circ \partial_n^{(i)} \end{array}$$

map $\mathrm{id}_k \mapsto \mathrm{id}_k$ by the uniqueness of an analytic morphism $\Delta^{n+1} \to *$ and $[0,1]^{n+1} \to \mathscr{M}(k)$. It implies that any morphism in $\mathrm{Hom}([0,1]^{n+1}, \mathscr{M}(k))$ is degenerate, and one obtains

$$\mathrm{C}_*^{\triangle}(\mathscr{M}(k)) = (0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots)$$

and

$$\begin{array}{ccccccccccc}
0 & = & (0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots) \\
\downarrow & & \| & & \| & & \downarrow & & \downarrow & & \\
\mathrm{D}^{\square}_{*}(\mathscr{M}(k)) & = & (0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots) \\
\downarrow & & \| & & \downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \\
\mathrm{Q}^{\square}_{*}(\mathscr{M}(k)) & = & (0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots) \\
\downarrow & & \| & & {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{C}^{\square}_{*}(\mathscr{M}(k)) & = & (0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots) \\
\downarrow & & \| & & \downarrow & & \| & & \| & & \\
0 & = & (0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots).
\end{array}$$

Thus one has

$$\mathrm{H}_n(\mathscr{M}(k)) = \begin{cases} \mathbb{Z} & (n = 0) \\ 0 & (n > 0) \end{cases} .$$

□

Proposition 3.2.5 (homotopy invariance)**.** *Let X be a k-$\mathscr{A}$ space. Then there exists a canonical isomorphism*

$$\begin{aligned} & H_n(X) \cong H_n(X \times_k \mathring{D}^1_k) \cong H_n(X \times_k D^1_k) \cong H_n(X \times_k \mathbb{A}^1_k) \\ \textit{resp.} \quad & H_n(X) \cong H_n(X \times_k \mathring{D}^{1\dagger}_k) \cong H_n(X \times_k D^{1\dagger}_k) \cong H_n(X \times_k \mathbb{A}^{1\dagger}_k) \end{aligned}$$

for each $n \in \mathbb{N}$, where $\mathring{D}^1_k$ (resp. $\mathring{D}^{1\dagger}_k$) is the open unit disc $\mathring{D}^1_k(1)$ (resp. $\mathring{D}^1_k(1)^\dagger$), and D^1_k (resp. $D^{1\dagger}_k$) is the closed unit disc $D^1_k(1)$ (resp. $D^1_k(1)^\dagger$).

Note that universality of the fibre product of the given forms holds with respect to a morphism from Δ^n or $[0,1]^n$ by Corollary 1.4.20. and Corollary 1.4.20.

Proof. We omit the dagger case. Let Y denote the k-analytic space $\mathring{\mathrm{D}}^1_k$, D^1_k, or $\mathbb{A}^1_k$. By the canonical projection $X \times_k Y \to X$, one has a homomorphisms $a_n \colon \mathrm{H}_n(X \times_k Y) \to \mathrm{H}_n(X)$. We verify that this is an isomorphism. The zero-section $X \to X \times_k Y$ induces the homomorphism $b_n \colon \mathrm{H}_n(X) \to \mathrm{H}_n(X \times_k Y)$, and since the composition $X \to X \times_k Y \to X$ is the identity, the corresponding composition $a_n \circ b_n \colon \mathrm{H}_n(X) \to \mathrm{H}_n(X \times_k Y) \to \mathrm{H}_n(X)$ is the identity. Consider the other composition $b_n \circ a_n \colon \mathrm{H}_n(X \times_k Y) \to \mathrm{H}_n(X) \to \mathrm{H}_n(X \times_k Y)$ corresponding to the composition $X \times_k \mathrm{D}^1_k \to X \to X \times_k Y$. It suffices to show that $b_n \circ a_n = \mathrm{id}$. Fix a power root system $\underline{p} \in E_{k,1}$ of p and set

$$g = \frac{1 - \underline{p}}{1 - p} \in k[E_{k,1}].$$

The non-cubical case: Denote by $g_i^{(l)}$ the the image of $g(t_i + \ldots + t_l) \in k[E_{k,l+1}]$ in k_{Δ^l} and by $g_{i,j}^{(l)}$ the image of $\partial_l^{(j)*}(g_i^{(l)}) \in k[E_{k,l}]$ in $k_{\Delta^{l-1}}$ for each $l \in \mathbb{N}_+$, $i = 1, \ldots, l$, and $j = 0, \ldots, l$. Formally set $g_{l+1}^{(l)} := 1$. Note that $g_{i,j}^{(l)} = g_{i-1}^{(l-1)}$ for each $j = 0, \ldots, i-1$, and $g_{i,j}^{(l)} = g_i^{(l-1)}$ for each $j = i, \ldots, n+1$.

Consider the case $Y = \mathrm{D}_k^1$ first. We prepare notation. Take an arbitrary morphism $\gamma \colon \Delta^n \to X \times_k \mathrm{D}_k^1$. The morphism γ induces two morphisms $\gamma' \colon \Delta^n \to X$ and $\gamma'' \colon \Delta^n \to \mathrm{D}_k^1$ by the canonical projections $X \times_k \mathrm{D}_k^1 \to X$ and $X \times_k \mathrm{D}_k^1 \to \mathrm{D}_k^1$. Thess correspondences induce group-homomorphisms $\mathrm{C}_n^{\triangle}(X \times_k \mathrm{D}_k^1) \to \mathrm{C}_n^{\triangle}(X) \colon \xi \mapsto \xi'$ and $\mathrm{C}_n^{\triangle}(X \times_k \mathrm{D}_k^1) \to \mathrm{C}_n^{\triangle}(X) \colon \xi \mapsto \xi''$. Let $0 \colon \Delta^n \to \mathrm{D}_k^1$ be the constant morphism onto the k-rational point $0 \in \mathrm{D}_k^1(k)$. The given morphism γ coincides with $\gamma' \times_k \gamma''$, and the morphism correponding to the image $(b \circ a)[\gamma] \in \mathrm{C}_n^{\triangle}(X \times_k \mathrm{D}_k^1)$ is the morphism $\gamma' \times_k 0$. By Lemma 1.5.1, the morphism γ'' is given by a bounded k-homomorphism $\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma'') \colon k\{T_1\} \to k_{\Delta^n}$. Set $f(\gamma)(t_0, \ldots, t_n) := \mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma'')(T_1) \in k_{\Delta^n}$. Consider the k-algebra homomorphism

$$\begin{array}{rcl} f(\gamma, i) \colon k[T_1] & \to & k_{\Delta^{n+1}} \\ T_1 & \mapsto & f(\gamma, i)(T_1)(t_0, \ldots, t_{n+1}) := g_{i+1}^{(n+1)} \sigma_{n+1}^{(i)*}(f(\gamma)) \end{array}$$

for each $i \in \mathbb{N}$ with $i \leq n+1$. Since $\|f(\gamma)\| = \|\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma'')(T_1)\| \leq \|T_1\| = 1$, one has

$$\|f(\gamma, i)(T_1)\| = \|g_{i+1}^{(n+1)} \sigma_{n+1}^{(i)*}(f(\gamma))\| \leq \|g_{i+1}^{(n+1)}\| \|\sigma_{n+1}^{(i)*}(f(\gamma))\| \leq 1 \times \|f(\gamma)\| \leq 1.$$

Therefore the homomorphism $f(\gamma, i) \colon k[T] \to k_{\Delta^{n+1}}$ is uniquely extended to a bounded k-homomorphism $f(\gamma, i) \colon k\{T\} \to k_{\Delta^{n+1}}$, and determines a morphism $\mathscr{M}(f(\gamma, i)) \colon \Delta^{n+1} \to \mathrm{D}_k^1$. Let $\gamma\{i, j\} \colon \Delta^{n+1} \to X \times_k \mathrm{D}_k^1$ denote the morphism

$$\gamma\{i, j\} = \gamma\{i, j\}' \times_k \gamma\{i, j\}'' := \gamma \circ \partial_n^{(j)} \circ \sigma_n^{(i)} \colon \Delta^{n+1} \xrightarrow{\sigma_n^{(i)}} \Delta^n \xrightarrow{\partial_n^{(j)}} \Delta^{n+1} \xrightarrow{\gamma} X \times_k \mathrm{D}_k^1$$

for each $i = 0, \ldots, n$ and $j = 0, \ldots, n+1$, and $\Gamma(\gamma)(t_0, \ldots, t_{n+1}) \in \mathrm{C}_{n+1}^{\triangle}(X)$ denote the singular simplex

$$\Gamma_n(\gamma) := \sum_{i=0}^{n} (-1)^i \left[(\gamma' \circ \sigma_{n+1}^{(i)}) \times_k \mathscr{M}(f(\gamma, i)) \right] = \sum_{i=0}^{n} (-1)^i \left[(\gamma' \circ \sigma_{n+1}^{(i)}) \times_k \mathscr{M}(g_{i+1}^{(n+1)} \sigma_{n+1}^{(i)*}(f(\gamma))) \right].$$

The correspondence $\gamma \to \Gamma_n(\gamma)$ determines a group-homomorphism $\Gamma_n \colon \mathrm{C}_n^{\triangle}(X) \to \mathrm{C}_{n+1}^{\triangle}(X) \colon \xi \mapsto \Gamma_n \xi$. One has

$$\begin{aligned} d_{n+1} \Gamma_n[\gamma] &= d_{n+1}[\Gamma_n(\gamma)] = \sum_{i=0}^{n} (-1)^i d_{n+1} \left[(\gamma' \circ \sigma_{n+1}^{(i)}) \times_k \mathscr{M}(f(\gamma, i)) \right] \\ &= \sum_{i=0}^{n} (-1)^i \left(\sum_{j=0}^{n+1} (-1)^j \left[(\gamma' \circ \sigma_{n+1}^{(i)} \circ \partial_{n+1}^{(j)}) \times_k \mathscr{M}(\partial_{n+1}^{(j)*}(g_{i+1,j}^{(n+1)})(\sigma_{n+1}^{(i)} \circ \partial_{n+1}^{(j)})^*(f(\gamma))) \right] \right) \\ &= \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} \left[\gamma\{i-1, j\}' \times_k \mathscr{M}(g_i^{(n)} \mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma\{i, j\}'')) \right] \end{aligned}$$

$$
\begin{aligned}
& + \sum_{i=0}^{n}(-1)^{2i}\left[\gamma' \times_k \mathscr{M}(g_i^{(n)} f(\gamma))\right] + \sum_{i=0}^{n}(-1)^{2i+1}\left[\gamma' \times_k \mathscr{M}(g_{i+1}^{(n)} f(\gamma))\right] \\
& + \sum_{i=0}^{n-1}\sum_{j=i+2}^{n+1}(-1)^{i+j}\left[\gamma\{i, j-1\}' \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma\{i, j-1\}''))\right] \\
= \; & \sum_{i=0}^{n-1}\sum_{j=0}^{i}(-1)^{i+1+j}\left[\gamma\{i, j\}' \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma\{i, j\}''))\right] \\
& + \sum_{i=0}^{n}\left[\gamma' \times_k \mathscr{M}(g_i^{(n)} f(\gamma))\right] - \sum_{i=0}^{n}\left[\gamma' \times_k \mathscr{M}(g_{i+1}^{(n)} f(\gamma''))\right] \\
& + \sum_{i=0}^{n-1}\sum_{j=i+1}^{n}(-1)^{i+j+1}\left[\gamma\{i, j\}' \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma\{i, j\}''))\right] \\
= \; & \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j+1}\left[\gamma\{i, j\}' \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma\{i, j\}''))\right] \\
& + \left[\gamma' \times_k \mathscr{M}(g_0^{(n)} f(\gamma))\right] - \left[\gamma' \times_k \mathscr{M}(g_{n+1}^{(n)} f(\gamma))\right] \\
= \; & \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j+1}\left[\gamma\{i, j\}' \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma\{i, j\}''))\right] + [\gamma] - [\gamma' \times_k 0].
\end{aligned}
$$

On the other hand, one calculates

$$
\begin{aligned}
\Gamma_{n-1} d_n[\gamma] &= \Gamma_{n-1}\sum_{j=0}^{n}(-1)^j\left[\gamma \circ \partial_n^{(j)}\right] = \sum_{j=0}^{n}(-1)^j \Gamma_{n-1}(\gamma \circ \partial_n^{(j)}) \\
&= \sum_{j=0}^{n}(-1)^j\sum_{i=0}^{n-1}(-1)^i\left[((\gamma \circ \partial_n^{(j)})' \circ \sigma_n^{(i)}) \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)((\gamma \circ \partial_n^{(j)})'' \circ \sigma_n^{(i)}))\right] \\
&= \sum_{j=0}^{n}(-1)^j\sum_{i=0}^{n-1}(-1)^i\left[\gamma' \circ \partial_n^{(j)} \circ \sigma_n^{(i)}) \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma'' \circ \partial_n^{(j)} \circ \sigma_n^{(i)}))\right] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}\left[\gamma\{i, j\}' \times_k \mathscr{M}(g_{i+1}^{(n)}\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma\{i, j\}''))\right]
\end{aligned}
$$

and it follows

$$
d_{n+1}\Gamma_n[\gamma] + \Gamma_{n-1}d_n[\gamma] = [\gamma] - [\gamma' \times_k 0] = [\gamma] - (b \circ a)[\gamma].
$$

Now take an arbitrary cycle $\xi \in \mathrm{C}_n^{\triangle}(X \times_k \mathrm{D}_k^1)$. By the calculation above, one has

$$
\xi - (b \circ a)\xi = d_{n+1}\Gamma_n\xi + \Gamma_{n-1}d_n\xi = d_{n+1}\Gamma_n\xi \in d_{n+1}\mathrm{C}_{n+1}^{\triangle}(X \times_k \mathrm{D}_k^1),
$$

and hence the induced homomorphism $b \circ a\colon \mathrm{H}_n^{\triangle}(X \times_k \mathrm{D}_k^1) \to \mathrm{H}_n^{\triangle}(X \times_k \mathrm{D}_k^1)$ is the identity. We conclude that the homomorphisms $a\colon \mathrm{H}_n^{\triangle}(X \times_k \mathrm{D}_k^1) \to \mathrm{H}_n^{\triangle}(X)$ and $b\colon \mathrm{H}_n^{\triangle}(X) \to \mathrm{H}_n^{\triangle}(X \times_k \mathrm{D}_k^1)$ is isomorphisms which are the inverse of the other.

Now consider the case Y is $\mathring{\mathrm{D}}^1_k$ or $\mathbb{A}^1_k$. We have only to show that the composition $b \circ a \colon \mathrm{H}^{\triangle}_n(X \times_k Y) \to \mathrm{H}^{\triangle}_n(X \times_k \mathrm{D}^1_k)$ is the identity. Take an arbitrary cycle $\xi \in \mathrm{C}^{\triangle}_n(X \times_k Y)$, and a presentation of ξ as a formal sum $a_1[\gamma_1] + \cdots + a_l[\gamma_l]$ by some $l \in \mathbb{N}$, $a_1, \ldots, a_l \in \mathbb{Z}$, and $\gamma_1, \ldots, \gamma_l \in \mathrm{Hom}(\Delta^n, X \times_k Y)$. Since $\Delta^n_{\overline{k}}$ is compact, there exists some $d \in (0, \infty)$ such that $\mathrm{D}^1_k(d) = \mu(k\{d^{-1}T\}) \subset Y$ and the image $\gamma_i^{\#}(\Delta^n_{\overline{k}})$ is contained in $X \times_k \mathrm{D}^1_k(d)$ for each $i = 1, \ldots, l$. Since k is of mixed characteristic, the valuation of k is non-trivial. Therefore we may assume and do $\mathrm{D}^1_k(d)$ is a strict k-dagger space replacing d to a suitable parametre in $|k^{\times}|$. By the universality of a Weierstrass domain, Proposition 1.2.19, the morphisms γ_i uniquely factors through $X \times_k \mathrm{D}^1_k(d)$. Denote by

$$\gamma_i|^{X \times_k \mathrm{D}^1_k(d)} \colon \Delta^n \to X \times \mathrm{D}^1_k(d)$$

the restriction of γ_i for each $i = 1, \ldots, l$, and set

$$\xi|^{X \times_k \mathrm{D}^1_k(d)} := a_1 \left[\gamma_1|^{X \times_k \mathrm{D}^1_k(d)}\right] + \cdots + a_l \left[\gamma_l|^{X \times_k \mathrm{D}^1_k(d)}\right] \in \mathrm{C}^{\triangle}_n(X \times_k \mathrm{D}^1_k(d)).$$

Then the image of it by the homomorphism $\mathrm{C}^{\triangle}_n(X \times_k \mathrm{D}^1_k(d)) \to \mathrm{C}^{\triangle}_n(X \times_k Y)$ induced by the inclusion $X \times_k \mathrm{D}^1_k(d) \to X \times_k Y$ coincides wiith ξ. Since $X \times_k \mathrm{D}^1_k(d)$ is isomorphic to $X \times_k \mathrm{D}^1_k$ by the strictness of $\mathrm{D}^1_k(d)$, we know that $b \circ a \colon \mathrm{H}^{\triangle}_n(X \times_k \mathrm{D}^1_k(d)) \to \mathrm{H}^{\triangle}_n(X \times_k \mathrm{D}^1_k(d))$ is the identity. Therefore there exists an element $\eta \in \mathrm{C}^{\triangle}_{n+1}(X \times_k \mathrm{D}^1_k(d))$ such that

$$\left(\xi|^{X \times_k \mathrm{D}^1_k(d)}\right) - (b \circ a)\left(\xi|^{X \times_k \mathrm{D}^1_k(d)}\right) = d_{n+1}\eta.$$

Let

$$\eta|^{X \times_k Y} \in \mathrm{C}^{\triangle}_{n+1}(X \times_k Y).$$

be the image of η by the homomorphism $\mathrm{C}^{\triangle}_{n+1}(X \times_k \mathrm{D}^1_k(d)) \to \mathrm{C}^{\triangle}_{n+1}(X \times_k Y)$ induced by the inclusion $X \times_k \mathrm{D}^1_k(d) \to X \times_k Y$, and then one has

$$d_{n+1}\left(\eta|^{X \times_k Y}\right) = (d_{n+1}\eta)|^{X \times_k Y} = \left(\left(\xi|^{X \times_k \mathrm{D}^1_k(d)}\right) - (b \circ a)\left(\xi|^{X \times_k \mathrm{D}^1_k(d)}\right)\right)\Big|^{X \times_k Y} = \xi - (b \circ a)\xi.$$

It follows that the images of ξ and $(b \circ a)\xi$ in $\mathrm{H}^{\triangle}_n(X \times_k Y)$ coincides with each ather, and we conclude the homomorphism $b \circ a \colon \mathrm{H}^{\triangle}_n(X \times_k Y) \to \mathrm{H}^{\triangle}_n(X \times_k Y)$ is the identity.

The cubical case: Take an arbitrary cycle $\gamma \in \mathrm{C}_n(X \times_k Y)$, and set $\gamma = z_1[\gamma_1] + \cdots + z_m[\gamma_m]$ for some $m \in \mathbb{N}$, $z_1, \ldots, z_m \in \mathbb{Z}$, and $\gamma_1, \ldots, \gamma_m \in \mathrm{Hom}([0,1]^n, X \times_k Y)$. Denote by c_n the homomorphism $\mathrm{Q}^{\square}_n(X \times_k Y) \to \mathrm{Q}^{\square}_{n+1}(X \times_k Y)$ given by setting

$$c_n[\xi](t_1, \ldots, t_{n+1}) := \left[\xi'(t_1, \ldots, t_n) \times_k \mathscr{M}(\mathrm{H}^0(\cdot, \mathbb{G}_a)(\xi'')(t_1, \ldots, t_n)g(t_{n+1}))\right] \in \mathrm{Q}^{\square}_{n+1}(X \times_k Y)$$

for each morphism $\xi \colon [0,1]^n \to X \times_k Y$. Let e_n be the homomorphism $\mathrm{Q}^{\square}_n(X) \to \mathrm{Q}^{\square}_{n+1}(X)$ induced by the canonical projection $[0,1]^{n+1} \to [0,1]^n \colon (t_1, \ldots, t_{n+1}) \mapsto (t_1, \ldots, t_n)$. By the corresponding corollary of Lemma 1.5.1, the morphisms $\gamma''_1, \ldots, \gamma''_m \in \mathrm{Hom}([0,1]^n, Y)$ is determined by analytic functions $f_1, \ldots, f_m \in k_{[0,1]^n}$ with suitable norms. Denote by

$\Gamma_i \in \mathrm{Hom}([0,1]^{n+1}, Y)$ the fibre product $(e_n \circ a_n)(\gamma_i) \times_k \mathscr{M}(f_i(t_1, \ldots, t_n)g(t_{n+1}))$ for each $i = 1, \ldots, m$. One calculates

$$
\begin{aligned}
& d[\Gamma_i] = \sum_{j=1}^{n+1} \sum_{\sigma=0}^{1} (-1)^{j+\sigma} [\Gamma_i|_{t_j=\sigma}] \\
= \ & \sum_{j=1}^{n} \sum_{\sigma=0}^{1} (-1)^{j+\sigma} \left[((e_n \circ a_n)(\gamma_i))|_{t_j=\sigma}) \times_k \mathscr{M}(f_i(t_{j,\sigma})g(t_n)) \right] + (-1)^{n+1}((b_n \circ a_n)[\gamma] - [\gamma]) \\
= \ & \sum_{j=1}^{n} \sum_{\sigma=0}^{1} (-1)^{j+\sigma} \left[\gamma_i(t_{j,\sigma})' \times_k \mathscr{M}(\mathrm{H}^0(\cdot, \mathbb{G}_a)(\gamma_i'')(t_{j,\sigma})g(t_n)) \right] + (-1)^{n+1}((b_n \circ a_n)[\gamma] - [\gamma]) \\
= \ & \sum_{j=1}^{n} \sum_{\sigma=0}^{1} (-1)^{j+\sigma} c_{n-1}([\gamma_i|_{t_j=\sigma}]) + (-1)^{n+1}((b_n \circ a_n)[\gamma] - [\gamma])
\end{aligned}
$$

for each $i = 1, \ldots, m$, where $t_{j,\sigma} := (t_1, \ldots, t_{j-1}, \sigma, t_j, \ldots, t_{n-1})$, and therefore

$$
\begin{aligned}
& d \sum_{i=1}^{m} (-1)^{n+1} z_i [\Gamma_i] = \sum_{i=1}^{n} (-1)^{n+1} z_i d[\Gamma_i] \\
= \ & \sum_{i=1}^{m} z_i \sum_{j=1}^{n} \sum_{\sigma=0}^{1} (-1)^{n+1+j+\sigma} c_{n-1}([\gamma_i|_{t_j=\sigma}]) + \sum_{i=1}^{m} z_i((b \circ a)[\gamma_i] - [\gamma_i]) \\
= \ & c_{n-1}\left(d \sum_{i=1}^{m} z_i[\gamma_i] \right) + ((b \circ a) - 1)\left(\sum_{i=1}^{m} z_i[\gamma_i] \right) \\
= \ & (b \circ a)(\gamma) - (\gamma)
\end{aligned}
$$

because γ is a cycle. We conclude that the homomorphism $b \circ a \colon \mathrm{H}_n^{\square}(X \times_k Y) \to \mathrm{H}_n^{\square}(X \times_k Y)$ is the identity. □

Corollary 3.2.6. *For integers $n, m \in \mathbb{N}$ and $d \in (0, \infty)^m$, one has*

$$
\begin{aligned}
H_n(\mathring{D}_k^m(d)) \ & \cong_{\mathbb{Z}} \ H_n(D_k^m(d)) \cong_{\mathbb{Z}} H_n(\mathbb{A}_k^m) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & (n = 0) \\ 0 & (n > 0) \end{cases} \\
H_n(\mathring{D}_k^m(d)^\dagger) \ & \cong_{\mathbb{Z}} \ H_n(D_k^m(d)^\dagger) \cong_{\mathbb{Z}} H_n(\mathbb{A}_k^{m\dagger}) \cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & (n = 0) \\ 0 & (n > 0) \end{cases} .
\end{aligned}
$$

Proof. It directly follows from the dimension axiom and the homotopy invariance. □

Before proving the existence of Mayer-Vietoris exact sequence, we have to justify the barycentric subdivision of a cycle. The proof of the existence of a homotopy between the barycentric subdivision operator and the identity totally deffers depending on whether one considers the singular homology or the cubical singular homology. We prove it first for the singular homology. To tell the truth, the cubical singular homology does not satisfies the existence of such a homoytopy because cubical the singular chain complex

has insufficient boundaries. In order to examine this fact, we will prove the existence of a homotopy for what is called the "generalised" cubical singular homology, which is the homology of the "general" cubical singular chain contaning the cubical singular chain. The construction of the homotopy needs "general" cubical singular simplices. Well, what is a "general" one? Recall that we are regarding exponential-like maps $x(t) \in \mathbb{Q}^{\vee}$ as analytic functions on $[0,1]$. Moreover for a general dimension $n \in \mathbb{N}$, we deal with formal elements $x(a_1 t_1 + \cdots + a_n t_n + c)$ given as exponential-like functions $x(t) \in \mathbb{Q}_k^{\vee}$ of which the variable t is substituted by a $\mathbb{Z}$-linear expression $t = a_1 t_1 + \cdots + a_n t_n + c$ of n-variables $t_1, \ldots, t_n$. This definition allows one to substitute a $\mathbb{Z}$-linear expressions $t_i = a_{i,1} s_1 + \cdots + a_{i,m} s_m + c_i$ for the variables $t_1, \ldots, t_n$ of an arbitrary analytic function $f(t_1, \ldots, t_n)$ on $[0,1]^n$ and to obtain the analytic function $f(t_1(s_1, \ldots, s_m), \ldots, t_n(s_1, \ldots, s_m))$ on $[0,1]^m$. This is the pull-back by an integral affine map. If k is algebraically closed, then the pull-back by a (not necessarily integral) affine map is also valid. However, one is not allowed to substitute polynomials for the n-variables $t_1, \ldots, t_n$ of an analytic function on $[0,1]^n$ because of the lack of a function of the form $x(t_1 t_2)$ for $x \in \mathbb{Q}^{\vee}$. The construction of a homotopy between the barycentric subdivision operator and the identity needs the pull-back of a function by a coordinate change by polynomials which is not an affine map in general, and that is why we do not have such a homotopy for the cubical singular homology. A "general" analytic function should be defined so that it admits the pull-back by a coordinate change by polynomials. See Lemma 6.1.1 in Appendix the construction of such a homotopy for a "general" analytic cubical singular homology.

Definition 3.2.7. *For an integer $n \in \mathbb{N}$, denote by $[n] \subset \mathbb{N}$ the subset $\{0, \ldots, n-1\}$. In particular, $[0] = \emptyset$.*

Definition 3.2.8. *For integers $n \in \mathbb{N}$ and $i \in [n+1]$, and for an injective map $\sigma \colon [i] \hookrightarrow [n+1]$, denote by $a_{\sigma}^{(n,i)} = (a_{\sigma,0}^{(n,i)}, \ldots, a_{\sigma,n}^{(n,i)}) \in \mathbb{Q}^{n+1}$ the vector whose entry $a_{\sigma,j}^{(n,i)}$ is $(n+1-i)^{-1}$ for any $j \in [n+1] \backslash \sigma([i])$ and is 0 for any $j \in \sigma([i])$. In particular, $a_{id}^{(0,0)} = 1 \in \mathbb{Q}$. For integers $n \in \mathbb{N}$ and $i, j \in [n+2]$, and for an injective map $\sigma \colon [i] \hookrightarrow [n+1]$, denote by $b_{\sigma,j}^{(n,i)} = (b_{\sigma,j,0}^{(n,i)}, \ldots, b_{\sigma,j,n}^{(n,i)}) \in \mathbb{Q}^{n+1}$ the vector defined as follows: If $j \in [i]$, then $b_{\sigma,j}^{(n,i)} := a_{\sigma}^{(n,j)}$. If $j \in [n+2] \backslash [i]$, then let $m_{\sigma,j} \in \mathbb{N}$ be the $(j-i)$-th least element of $[n+2] \backslash \sigma([i])$, and define $b_{\sigma,j}^{(n,i)}$ as the vector whose entry $b_{\sigma,j,k}^{(n,i)}$ is 1 for $k = m_{\sigma,j}$ and is 0 for any $k \in \{0, \ldots, n\} \backslash \{m_{\sigma,j}\}$. In particular, $b_{id,0}^{(0,0)} = 1 \in \mathbb{Q}$. For integers $n \in \mathbb{N}$, $i \in [n+1]$, and $j \in [n+2] \backslash [i+1]$, and for an injective map $\sigma \colon [i] \hookrightarrow [n+1]$, denote by $\sigma \sqcup (j-i) \colon [i+1] \hookrightarrow [n+1]$ the injective map determined by $(\sigma \sqcup (j-i))|_{[i]} = \sigma$ and $(\sigma \sqcup (j-i))(i) = m_{\sigma,j}$. For integers $n \in \mathbb{N}$, $i \in [n]$, and $k \in [n+1]$, and for an injective map $\sigma \colon [i] \hookrightarrow [n]$, denote by $k \sqcup \sigma \colon [i+1] \hookrightarrow [n+1]$ the injective map determined by $(k \sqcup \sigma)(0) = k$, $(k \sqcup \sigma)(l) = \sigma(l-1)$ for any $l \in [i+1] \backslash \{0\}$ such that $\sigma(l-1) < k$, and $(k \sqcup \sigma)(l) = \sigma(l-1)+1$ for any $l \in [i+1] \backslash \{0\}$ such that $\sigma(l-1) \geq k$. For integers $n \in \mathbb{N}$ and $i \in [n]$, and for an injective map $\sigma \colon [i] \hookrightarrow [n+1]$, define $\Sigma^{(n,i)}(\sigma) \in \mathbb{N}$ inductively on i by the following way: When $i = 0$, then set $\Sigma^{(n,i)}(\sigma) := 0$. When $i > 0$, assume $\Sigma^{(n,i-1)}$ has already been defined. Set $\Sigma^{(n,i)}(\sigma) := \Sigma^{(n,i-1)}(\sigma|_{[i-1]}) + l_{\sigma|_{[i-1]},\sigma(i-1)}$, where $l_{\sigma|_{[i-1]},\sigma(i-1)} \in \mathbb{N}$ is the number of integers in the set $([n] \backslash \sigma([i-1])) \cap [\sigma(i-1)]$.*

Definition 3.2.9. *Suppose k is algebraically closed. Let X be a k-$\mathscr{A}$ space, $n \in \mathbb{N}$, and $f(t_0, \ldots, t_n) \colon \Delta^n \to X$ a morphism. Define $bf \in C_n^{\triangle}(X)$ as follws:*

$$Bf \coloneqq \sum_{\sigma \in Aut([n+1])} (-1)^{\Sigma^{(n,n+1)}(\sigma)} \left[f\left(\sum_{i=0}^{n} a^{(n,i)}_{\sigma|_{[i]}} t_i \right) \right] = \sum_{\sigma \in Aut([n+1])} (-1)^{\Sigma^{(n,n+1)}(\sigma)} \left[f \begin{pmatrix} \sum_{i=0}^n a^{(n,i)}_{\sigma|_{[i]},0} t_i \\ \vdots \\ \sum_{i=0}^n a^{(n,i)}_{\sigma|_{[i]},n} t_i \end{pmatrix} \right].$$

In particular when $n = 0$, then $bf = [f]$. It induces a group homomophism

$$\begin{array}{rcl} B \colon C_n^{\triangle}(X) & \to & C_n^{\triangle}(X) \\ [f] & \mapsto & B[f] \coloneqq Bf \end{array}$$

identifying $C_n^{\triangle}(X)$ as the free $\mathbb{Z}$-module generated by non-degenerate paths.

Lemma 3.2.10 (barycentric subdivision for simplices)**.** *In the situation above, one has*

$$\overline{\xi} = \overline{B\xi} \in H_n^{\triangle}(X)$$

for any $\xi \in \ker d_n$.

The proof is totally algebraic and quite formal, and is done in much the same way as that of the corresponding fact in algebraic topology in [HAT]. Note that if one deals with the geometric analytic singular homology instead of the analytic singular homology and if one replaces the condition "great" to "universally great", he or she may remove the assumption that k is algebraically closed. The reason we assumed the algebraic closedness is because we used an affine map which is not integral in the definition of the barycentric subdivision operator B.

Proof. We construct a homotopy $\Phi \colon \mathrm{C}_*^{\triangle}(X) \to \mathrm{C}_{*+1}^{\triangle}(X)$ between b and id. For each $n \in \mathbb{N}$ and $f \colon \Delta^n \to X$, define $\Phi f \in \mathrm{C}_{n+1}^{\triangle}(X)$ as follows:

$$\begin{aligned} \Phi f &\coloneqq (-1)^n \sum_{i=0}^{n} \sum_{\sigma \colon [i] \hookrightarrow [n+1]} (-1)^{i+\Sigma^{(n,i)}(\sigma)} \left[f\left(\sum_{j=0}^{n+1} b^{(n,i)}_{\sigma,j} t_j \right) \right] \\ &= \sum_{i=0}^{n} \sum_{\sigma \colon [i] \hookrightarrow [n+1]} (-1)^{i+\Sigma^{(n,i)}(\sigma)} \left[f \begin{pmatrix} \sum_{j=0}^n b^{(n,i)}_{\sigma,j,0} t_j \\ \vdots \\ \sum_{j=0}^n b^{(n,i)}_{\sigma,j,n} t_j \end{pmatrix} \right]. \end{aligned}$$

It induces a group homomophism

$$\begin{array}{rcl} \Phi \colon \mathrm{C}_n^{\triangle}(X) & \to & \mathrm{C}_{n+1}^{\triangle}(X) \\ [f] & \mapsto & \Phi[f] \coloneqq \Phi f \end{array}$$

for each $n \in \mathbb{N}$ identifying $\mathrm{C}_n^{\triangle}(X)$ as the free $\mathbb{Z}$-module generated by non-degenerate paths. Take any $n \in \mathbb{N}$ and $f \coloneqq \Delta^n \to X$. We try brief calculations. Be careful that we regard $\mathrm{C}_n^{\triangle}(X)$ as a $\mathbb{Z}$-submodule of $\mathrm{C}_n^{\triangle}(X, \mathbb{Q})$ by the canonical embedding.

$$\partial \left[f\left(\sum_{j=0}^{n} b^{(n,i)}_{\sigma,j} t_j \right) \right]$$

$$
\begin{aligned}
= \;& \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] + \sum_{k=1}^{i-1} (-1)^k \left[f\left(\sum_{j=0}^{k-1} b^{(n,i)}_{\sigma,j} t_j + \sum_{j=k}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
& + (-1)^i \left[f\left(\sum_{j=0}^{i-1} b^{(n,i)}_{\sigma,j} t_j + \sum_{j=i}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] + \sum_{k=i+1}^{n+1} (-1)^k \left[f\left(\sum_{j=0}^{k-1} b^{(n,i)}_{\sigma,j} t_j + \sum_{j=k}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
= \;& \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
& + 2^{-1} \sum_{k=1}^{i-1} (-1)^k \left(\left[f\left(\sum_{j=0}^{k-1} b^{(n,i)}_{\sigma,j} t_j + \sum_{j=k}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] - \left[f\left(\sum_{j=0}^{k-1} b^{(n,i)}_{\sigma\circ(k,k+1),j} t_j + \sum_{j=k}^{n-1} b^{(n,i)}_{\sigma\circ(k,k+1),j+1} t_j \right) \right] \right) \\
& + (-1)^i \left[f\left(\sum_{j=0}^{i-1} b^{(n,i-1)}_{\sigma|_{[i]},j} t_j + \sum_{j=i}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] + \sum_{k=i+1}^{n+1} (-1)^k \left[f\left(\sum_{j=0}^{k-1} b^{(n,i)}_{\sigma,j} t_j + \sum_{j=k}^{n-1} b^{(n,i+1)}_{\sigma\sqcup(k-i),j+1} t_j \right) \right] \\
= \;& \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] + 0 + (-1)^i \left[f\left(\sum_{j=0}^{i-1} b^{(n,i-1)}_{\sigma|_{[i]},j} t_j + \sum_{j=i}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
& + \sum_{k=i+1}^{n+1} (-1)^k \left[f\left(\sum_{j=0}^{k-1} b^{(n,i)}_{\sigma,j} t_j + \sum_{j=k}^{n-1} b^{(n,i+1)}_{\sigma\sqcup(k-i),j+1} t_j \right) \right] \\
= \;& \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] + (-1)^i \left[f\left(\sum_{j=0}^{i-1} b^{(n,i-1)}_{\sigma|_{[i]},j} t_j + \sum_{j=i}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
& + \sum_{k=i+1}^{n+1} (-1)^k \left[f\left(\sum_{j=i+1}^{k-1} b^{(n,i)}_{(\sigma\sqcup(k-i))|_{[i+1]},j} t_j + \sum_{j=k}^{n-1} b^{(n,i+1)}_{\sigma\sqcup(k-i),j+1} t_j \right) \right] \\
\therefore \;\; & \partial(\Phi f) = \sum_{i=0}^{n} \sum_{\sigma:\,[i]\hookrightarrow[n+1]} (-1)^{n+i+\Sigma^{(n,i)}(\sigma)} \partial \left[f\left(\sum_{j=0}^{n} b^{(n,i)}_{\sigma,j} t_j \right) \right] \\
= \;& \sum_{i=1}^{n} \sum_{\sigma:\,[i]\hookrightarrow[n+1]} (-1)^{n+i+\Sigma^{(n,i)}(\sigma)} \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
& + \sum_{i=0}^{n} \sum_{\sigma:\,[i]\hookrightarrow[n+1]} (-1)^{n+\Sigma^{(n,i)}(\sigma)} \left[f\left(\sum_{j=0}^{i-1} b^{(n,i-1)}_{\sigma|_{[i]},j} t_j + \sum_{j=i}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
& + \sum_{i=0}^{n} \sum_{\sigma:\,[i]\hookrightarrow[n+1]} \sum_{k=i+1}^{n+1} (-1)^{n+\Sigma^{(n,i+1)+1}(\sigma\sqcup(k-i))} \left[f\left(\sum_{j=0}^{k-1} b^{(n,i)}_{(\sigma\sqcup(k-i))|_{[i+1]},j} t_j + \sum_{j=k}^{n-1} b^{(n,i+1)}_{\sigma\sqcup(k-i),j+1} t_j \right) \right] \\
= \;& \sum_{i=1}^{n} \sum_{\sigma:\,[i]\hookrightarrow[n+1]} (-1)^{n+i+\Sigma^{(n,i)}(\sigma)} \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
& + (-1)^n \left[f\left(\sum_{j=0}^{n-1} b^{(n,0)}_{\sigma,j+1} t_j \right) \right] - \sum_{\sigma:\,[n+1]\hookrightarrow[n+1]} (-1)^{n+\Sigma^{(n,n+1)}(\sigma)} \left[f\left(\sum_{j=0}^{n} b^{(n,n)}_{\sigma,j} t_j \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{\sigma\colon [i]\hookrightarrow[n+1]} (-1)^{n+i+\Sigma^{(n,i)}\sigma} \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
&\quad +(-1)^n [f(t_0,\ldots,t_n)] + \sum_{\sigma\colon [n+1]\hookrightarrow[n+1]} (-1)^{n+\Sigma^{(n,n+1)}(\sigma)} \left[f\left(\sum_{j=0}^{n} a^{(n,j)}_{\sigma,j} t_j \right) \right] \\
&= \sum_{i=1}^{n} \sum_{\sigma\colon [i]\hookrightarrow[n+1]} (-1)^{n+i+\Sigma^{(n,i)}\sigma} \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] + (-1)^n([f] - B[f]) \\
&\quad \Phi(\partial f) = \sum_{i=0}^{n-1} \sum_{\sigma\colon [i]\hookrightarrow[n]} \sum_{k=0}^{n-1} (-1)^{n-1+i+\Sigma^{(n-1,i)}(\sigma)+k} \left[f \begin{pmatrix} \sum_{j=0}^{n-1} b^{(n-1,i)}_{\sigma,j,0} t_j \\ \vdots \\ \sum_{j=0}^{n-1} b^{(n-1,i)}_{\sigma,j,k-1} t_j \\ 0 \\ \sum_{j=0}^{n-1} b^{(n-1,i)}_{\sigma,j,k} t_j \\ \vdots \\ \sum_{j=0}^{n-1} b^{(n-1,i)}_{\sigma,j,n-1} t_j \end{pmatrix} \right] \\
&= \sum_{i=1}^{n} \sum_{\sigma\colon [i-1]\hookrightarrow[n]} \sum_{k=0}^{n-1} (-1)^{n-1+i-1+\Sigma^{(n,i)}(k\sqcup\sigma)} \left[f\left(\sum_{j=1}^{n} b^{(n,i)}_{k\sqcup\sigma,j} t_j \right) \right] \\
&= \sum_{i=1}^{n} \sum_{\sigma\colon [i]\hookrightarrow[n+1]} (-1)^{n+i+\Sigma^{(n,i)}(\sigma)} \left[f\left(\sum_{j=0}^{n-1} b^{(n,i)}_{\sigma,j+1} t_j \right) \right] \\
&\therefore\quad (\partial\Phi - \Phi\partial)[f] = (-1)^n([f] - B[f])
\end{aligned}
$$

□

Once barycentric subdivision is justified in our homology theory, one achieves Mayer-Vietoris exact sequence by a totally formal argument.

Proposition 3.2.11 (Mayer-Vietoris exact sequence)**.** *Suppose k is algebraically closed. Let X be a k-$\mathscr{A}$ and $U, V \subset X$ great domains, Definition 1.4.29, satisfying Int$(U/X) \cup$ Int$(V/X) = X$, where Int(U/X) and Int(V/X) are the relative interior of U and V in X. Then there exists a long exact sequence*

$$H^{\triangle}_{n+1}(X) \to H^{\triangle}_{n}(U \cap V) \to H^{\triangle}_{n}(U) \oplus H^{\triangle}_{n}(V) \to H^{\triangle}_{n}(X).$$

Even if k is not algebraically closed, if $U, V \subset X$ are universally great domains, then there exists a long exact sequence

$$H^{\triangle}_{n+1}(X/\overline{k}) \to H^{\triangle}_{n}((U \cap V)/\overline{k}) \to H^{\triangle}_{n}(U/\overline{k}) \oplus H^{\triangle}_{n}(V/\overline{k}) \to H^{\triangle}_{n}(X/\overline{k}).$$

Proof. We omit the geometric case. The exactness of the chains

$$0 \to \mathrm{C}^{\triangle}_{n}(U \cap V) \to \mathrm{C}^{\triangle}_{n}(U) \oplus \mathrm{C}^{\triangle}_{n}(V) \to \mathrm{C}_{n}(X).$$

is trivial, and hence we have only to prove that $\mathrm{C}^{\triangle}_n(X)$ is generated by the image of $\mathrm{C}^{\triangle}_n(U) \oplus \mathrm{C}_n(V)$ and $d_{n+1}\mathrm{C}^{\triangle}_{n+1}(X)$. Take an arbitrary $\gamma \in \mathrm{Hom}(\Delta^n, X)$. Since $\mathrm{Int}(U/X)$ and $\mathrm{Int}(V/X)$ cover X, they give an open covering of the closed subset $\Delta^n \subset \Delta^n_k$. Since Δ^n is a compact metric space, there exists a Lebesgue number $\lambda > 0$ of this covering. Dividing Δ^n into $(n!)^m$ pieces for sufficiently large $m \in \mathbb{N}$, by restricting γ on each pieces, which we identify with Δ^n by the translation by affine maps, we obtain finitely many morphisms from Δ^n each of whose image is contained in U or V because U and V are great domains. The difference from $\gamma \in \mathrm{C}^{\triangle}_n(X)$ itself and the sum of the pieces of γ, which are contained in the image of $\mathrm{C}^{\triangle}_n(U) \oplus \mathrm{C}^{\triangle}_n(V)$ belongs to $d_{n+1}\mathrm{C}^{\triangle}_{n+1}(X)$ by Lemma 3.2.10, and hence we have done. □

Corollary 3.2.12. *Let X be a k-$\mathscr{A}$ space and $U, V \subset X$ k-analytic domains which cover X. Then there exists a long exact sequence*

$$H^{\triangle}_{n+1}(C^{\triangle}_*(U) + C^{\triangle}_*(V)) \to H^{\triangle}_n(U \cap V) \to H^{\triangle}_n(U) \oplus H^{\triangle}_n(V) \to H^{\triangle}_n(C^{\triangle}_*(U) + C^{\triangle}_*(V)),$$

where $C^{\triangle}_*(U)+C^{\triangle}_*(V)$ is the image of $C^{\triangle}_*(U) \oplus C^{\triangle}_*(V)$ in $C^{\triangle}_*(X)$. In particular if the inclusion $C^{\triangle}_*(U) + C^{\triangle}_*(V) \hookrightarrow C^{\triangle}_*(X)$ is a quasi-isomorphism, then one has the Mayer-Vietoris exact sequence

$$H^{\triangle}_{n+1}(X) \to H^{\triangle}_n(U \cap V) \to H^{\triangle}_n(U) \oplus H^{\triangle}_n(V) \to H^{\triangle}_n(X).$$

Proof. It directly follows from the proof of the Mayer-Vietoris exact sequence. □

Corollary 3.2.13 (excision axiom)**.** *Sppose k is algebraically closed. Let $A, X' \subset X$ be open great domains satisfying $X = X' \cup A$ and set $A' := X' \cap A = A \backslash (X \backslash X')$. Then there exists a canonical isomorphism*

$$H^{\triangle}_n(X', A') \cong H^{\triangle}_n(X, A).$$

Even if k is not algebraically closed, if $A, X' \subset X$ be open universally great domains, then there exists a canonical isomorphism

$$H^{\triangle}_n(X'/\overline{k}, A') \cong H^{\triangle}_n(X/\overline{k}, A).$$

Proof. We omit the geometric case. By the long exact sequences of space pairs, we have a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{H}^{\triangle}_n(A) & \longrightarrow & \mathrm{H}^{\triangle}_n(X) & \longrightarrow & \mathrm{H}^{\triangle}_n(X, A) & \longrightarrow & \mathrm{H}^{\triangle}_{n-1}(A) & \longrightarrow & \mathrm{H}^{\triangle}_{n-1}(X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{H}^{\triangle}_n(A') & \longrightarrow & \mathrm{H}^{\triangle}_n(X') & \longrightarrow & \mathrm{H}^{\triangle}_n(X', A') & \longrightarrow & \mathrm{H}^{\triangle}_{n-1}(A') & \longrightarrow & \mathrm{H}^{\triangle}_{n-1}(X').
\end{array}$$

In addition, we have the Mayer-Vietoris exact sequence

$$\mathrm{H}^{\triangle}_n(A') \to \mathrm{H}^{\triangle}_n(A) \oplus \mathrm{H}^{\triangle}_n(X') \to \mathrm{H}^{\triangle}_n(X) \to \mathrm{H}^{\triangle}_{n-1}(A')$$

because $X = X' \cup A$ and $X' \cap A = A'$. We begin to prove that the centre column of the commutative diagram above is an isomorphism. Take an arbitrary element $[\xi_1] \in \mathrm{H}^{\triangle}_n(X, A)$, and let $\xi_1 \in \mathrm{C}^{\triangle}_n(X, A)$ be a representative of $[\xi_1]$. Take an $\eta_1 \in \mathrm{C}^{\triangle}_n(X)$ such that the image of η_1 in $\mathrm{C}^{\triangle}_n(X, A)$ is ξ_1, and there exist some $\eta_2 \in \mathrm{C}^{\triangle}_{n+1}(X)$, $\zeta_1 \in \mathrm{C}^{\triangle}_n(A)$, and $\eta'_1 \in \mathrm{C}^{\triangle}_n(X')$ such that $d_{n+1}\eta_2 + \zeta_1 + \eta'_1 = \eta_1$ in $\mathrm{C}^{\triangle}_n(X)$ by the proof of the Mayer-Vietoris exact sequence. Denote by $\xi'_1 \in \mathrm{C}^{\triangle}_n(X', A')$ the image of η'_1. Then in $\mathrm{C}^{\triangle}_{n-1}(X, A)$, we have $d_n\xi'_1 = d_n\eta'_1 = d_n\xi_1 - d_n(d_{n+1}\xi_2 + \zeta) = 0$ and hence $d_n\xi'_1 = 0$ also in $\mathrm{C}^{\triangle}_n(X', A')$ by the injectivity of $\mathrm{C}^{\triangle}_*(X', A') \to \mathrm{C}^{\triangle}_*(X, A)$. It follows that ξ'_1 is a cycle. Since $\xi_1 = \eta_1 = d_{n+1}\eta_2 + \zeta_1 + \eta'_1 = d_{n+1}\eta_2 + \zeta_1 + \xi'_1$ is a sum of cycles in $\mathrm{C}^{\triangle}_n(X, A)$, we have $[\xi_1] = [d_{n+1}\eta_2] + [\zeta_1] + [\xi'_1] = [\xi'_1]$ in $\mathrm{H}^{\triangle}_n(X, A)$. It implies $\mathrm{H}^{\triangle}_n(X', A') \to \mathrm{H}^{\triangle}_n(X, A)$ is surjective. On the other hand, take an arbitrary $[\xi'_1] \in \ker(\mathrm{H}^{\triangle}_n(X', A') \to \mathrm{H}^{\triangle}_n(X, A))$, and let $\xi'_1 \in \mathrm{C}^{\triangle}_n(X', A')$ is a representative of $[\xi'_1]$. Since $[\xi'_1] = 0$ in $\mathrm{H}^{\triangle}_n(X, A)$, there exists some $\xi_2 \in \mathrm{C}^{\triangle}_{n+1}(X, A)$ such that $d_{n+1}\xi_2 = \xi'_1$ in $\mathrm{C}^{\triangle}_n(X, A)$. Take an $\eta_2 \in \mathrm{C}^{\triangle}_{n+1}(X)$ such that the image of η_2 is ξ_2 in $\mathrm{C}^{\triangle}_{n+1}(X, A)$. By the proof of the Mayer-Vietoris exact sequence, there exist some $\eta_3 \in \mathrm{C}^{\triangle}_{n+2}(X)$, $\zeta_2 \in \mathrm{C}^{\triangle}_{n+1}(A)$, and $\eta'_2 \in \mathrm{C}^{\triangle}_{n+1}(X')$ such that the image of $d_{n+1}\eta_3 + \zeta_2 + \eta'_2$ in $\mathrm{C}^{\triangle}_n(X)$ coincides with η_2. Since $\xi'_1 = d_{n+1}\xi_2 = d_{n+1}\eta_2 = d_{n+1}(d_{n+2}\eta_3 + \zeta_2 + \eta'_2) = d_{n+1}\eta'_2$ in $\mathrm{C}^{\triangle}_n(X, A)$, we have $d_{n+1}\eta'_2 = \xi'_1$ in $\mathrm{C}^{\triangle}_n(X', A')$ because of the injectivity of the homomorphism $\mathrm{C}^{\triangle}_n(X', A') \to \mathrm{C}^{\triangle}_n(X, A)$ (from the snake lemma). It follows that ξ is a boundary in $\mathrm{C}^{\triangle}_n(X', A')$, i.e. $[\xi] = [d_{n+1}\eta_2] = 0$ in $\mathrm{H}^{\triangle}_n(X', A')$. Therefore the homomorphism $\mathrm{H}^{\triangle}_n(X', A') \to \mathrm{H}^{\triangle}_n(X, A)$ turns out to be injective. □

We have verified what is called the axiom of a homology theory. We have many other basic properties desired in a homology theory. We show that $\mathrm{H}_0(X)$ have a good relation with the arcwise connectedness of X at first. From now on, identify Δ^0 and Δ^1 as $[0, 1]^0$ and $[0, 1]$.

Lemma 3.2.14. *Let X be a k-$\mathscr{A}$ space. One has*

$$Hom([0, 1]^0, X) = X(k).$$

Proof. Trivial because $[0, 1]^0_k = \mathscr{M}(k)$. □

Definition 3.2.15. *Let X be a k-$\mathscr{A}$ space, and $x, y \in X(k)$ k-rational points. We say x and y share the analytically pathwise connected component and write $x \sim y$ if there exist some $m \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_m \in Hom([0, 1], X)$ such that $x = \gamma_1(0), \gamma_1(1) = \gamma_2(0), \ldots, \gamma_{m-1}(1) = \gamma_m(0), \gamma_m(1) = y$. The relation $\sim$ is an equivalence relation, and denote by $\pi_0(X)$ the collection of all equivalent class with respect to the relation $\sim$. X is said to be analytically pathwise connected if $\pi_0(X)$ consists of a single equivalent class.*

Note that if X is analytically pathwise connected, $\pi_0(X)$ is non-empty and hence X has a k-rational point. For example, for a non-trivial (not necessarily strict) topologically of finite type extension K/k, $\mathscr{M}(K)$ is not an analytically pathwise connected k-analytic space or an analytically pathwise connected k-dagger space, while it is analytically pathwise connected as both of a K-analytic space and a K-dagger space. Here we do not assume a topologically of finite type k-Banach algebra (or weakly complete k-algebra)

to be strict in the sense of [BER2]: we say a k-Banach algebra (resp. weakly complete k-algebra) A is said to be topologically of finite type is A is a general k-affinoid algebra (resp. a general k-dagger algebra), i.e. there is an admissible epimorphism from a general Tate algebra $k\{r^{-1}T\}$ (resp. a general Monsky-Washnitzer algebra $k\{r^{-1}T\}^{\dagger}$) onto A for some $r \in (0,\infty)^n$ for some $n \in \mathbb{N}$.

Definition 3.2.16. *Let X be a k-$\mathscr{A}$ space. X is said to be locally analytically pathwise connected if for any point $t \in X$ and any neighbourhood $U \in X$ of t, t has an analytically pathwise connected analytic neighbourhood in U.*

It is non-sense to consider this notion if k is not algebraically closed. Indeed, if k is not algebraically closed, a strict k-analytic space which does not consist of isolated points has a strict k-affinoid neighbourhood which is not a finite k-algebra. By Noether's normalisation lemma [BGR], 6.1.2/2, A is finite over some k-subalgebra isometrically isomorphic to a strict Tate algebra which is not k. Strict Tate algebra which is not k has a $\overline{k}$-valued maximal ideal which is not k-rational, and so does A by Going-up theorem. A maximal ideal is automatically closed by [BGR] 1.2.4/5, and hence $m \subset A$ determines the unique bounded valuation $x \in \mathscr{M}(A) \subset X$ on A whose support is m. Then x is a $\overline{k}$-rational point which is not k-rational. Fix an admissible epimorphism from a strict k-affinoid algebra $k\{T_1,\ldots,T_n\}$ onto A, and consider the corresponding closed embedding $\mathscr{M}(A) \hookrightarrow \mathrm{D}^n_k \subset \mathbb{A}^n_k$. Since k^n is closed in $\overline{k}$-ratinal points $\overline{k}^n = \mathbb{A}^n_k(\overline{k})$, there exists some open neighbourhood U of x in $\mathscr{M}(A) \subset \mathbb{A}^n_k$ such that U contains no k-rational point. Since $\mathscr{M}(A) \hookrightarrow X$ is a homeomorphism onto the image, there exists some open neighbourhood U' of x in X such that U' contains no k-rational point. Any neighourhood of t in U' has no k-rational point and hence is not analytically pathwise connected. Thus X is not locally analytically pathwise connected. We do not know there is an expamle of a non-strict k-analytic space which does not consist of isolated points and which has no k-rational point. Considering a general non-strict k-affinoid algebra is quite bothersome because of the lack of Noether's normalisation lemma.

Let X be a k-$\mathscr{A}$ space. For a k-rational point $t \in X(k)$, the analytically pathwise connected component of t in X is the topological subspace of X which is the union of all analytically pathwise connected analytic domain of X containing t. The set of k-rational points in an analytically pathwise connected component is an equivalent class with respect to the relation $\sim$, i.e. an element of $\pi_0(X)$, and hence any two analytically pathwise connected components do not intersect at a k-rational point. If X is locally analytically pathwise connected, then any analytically pathwise connected component of X is an open subspace. If k is algebraically closed and if X is strict, any non-empty open subspace has a k-rational point and hence analytically pathwise connected components are disjoint. Moreover the collection of analytically pathwise connected components cover X because any point of X has an analytically pathwise connected open subspace. On the other hand, if X is not locally analytically pathwise connected, the collection of analytically pathwise connected components does not necessarily cover X. There possibly exists some not k-rational point $t \in X$ such that there is no analytically pathwise connected analytic domain containing t. It is pity that there are many cases an analytic space is not locally analyti-

cally pathwise connected. For example, the spectrum $\mathscr{M}(K)$ of a non-trivial topologically of finite type extension K/k has no k-rational point, and hence is not locally analytically connected. The punctured unit disc $\mathrm{D}^1_k \backslash \{0,1\}$ is not locally analytically connected, either. Indeed, there is no analytically pathwise connected analytic domain containing the point $x \in \mathrm{D}^1_k \backslash \{0,1\} \subset \mathrm{D}^1_k$ corresponds to the norm of $k\{T_1\}$. The homotopy set $\pi_0(\mathrm{D}^1_k \backslash \{0,1\})$ is canonically bijective to the topological homotopy set $\pi_0^{\mathrm{top}}(\mathrm{D}^1_k \backslash \{0,1,x\})$. These are the most basic examples of not locally analytically pathwise connected analytic spaces. Note that the first example does not occur if k is algebraically closed and if we only consider a strict k-analytic space, and hence the important is the second one. See Proposition 5.1.6 for more detail.

Lemma 3.2.17. *Let X, Y be k-analytic spaces or k-dagger spaces, and $\phi \colon X \to Y$ a morphism. If X is analytically pathwise connected and if $\phi^\#$ is a surjective continuous map of underlying topological space, then Y is analytically pasthwise connected. On the other hand when ϕ is a G-local isomorphism, X is locally analytically pathwise connected if and only if Y is locally analytically pathwise connected.*

Proof. Just similar with the corresponding fact of pathwise connectedness in the topology theory. □

Proposition 3.2.18. *Let X be a k-$\mathscr{A}$ space. The canonical set-theoretical map*

$$X(k) = Hom([0,1]^0, X) = [Hom([0,1]^0, X)] \hookrightarrow C_0(X) = \ker d_0 \twoheadrightarrow H_0(X)$$

induces a set-theoretical map $\pi_0(X) \to H_0(X)$. Then the group-homomorphism $\mathbb{Z}^{\oplus \pi_0(X)} \to H_0(X)$ induced by the universality of the free Abelian group is an isomorphism. In particalur, X is analytically pathwise connected if and only if $H_0(X) \cong_{\mathbb{Z}} \mathbb{Z}$.

Proof. Trivial. □

Corollary 3.2.19. *Suppose k is algebraically closed. Let X be a strict k-analytic space or a strict k-dagger space. If $H_0(X) \cong_{\mathbb{Z}} \mathbb{Z}$, then X is arcwise connected.*

Proof. The condition $\mathrm{H}_0(X) \cong_{\mathbb{Z}} \mathbb{Z}$ guarantees that X is non-empty and analytically pathwise connected. Since k is algebraically closed and X is strict, any non-empty open subset of X has a k-rational point by the argument right after Definition 3.2.16. An analytic space and a dagger space are always locally connected by [BER1] 2.2.8, and it implies that each connected component of X has a k-rational pooint.

Since $[0,1]_k$ is connected by Lemma 1.2.3, any two k-rational points belonging to the same analytically pathwise connected component share the connected component in X because a continuous map preserves the connectedness. It follows that X is connected because each connected component of X has a k-rational point. Consequently X is arcwise connected because any connected k-analytic space and any connected k-dagger space are arcwise connected by [BER1], 3.2.1. Note that [BER1], 3.2.1. deals with only a good k-analytic space in the sense of [BER2], but it implies that a connected k-analytic space and a connected k-dagger space are locally arcwise connected and hence arcwise connected even if we do not assume goodness. □

The fundamental group of a topological group (or of an H-space) has two group structure. The one is the original group structure given to the fundamental group of a general topological space, which is induced by homotopically associative binomial connecting two loops. The other one is given in the following way: Let G be a topological group. The multiplication $*$ of G induce a associative binomial

$$\begin{array}{rcl} *\colon \mathrm{Hom}_{\mathrm{Tri}}(([0,1],0,1),(G,1,1))^2 & \to & \mathrm{Hom}_{\mathrm{Tri}}(([0,1],0,1),(G,1,1)) \\ (\gamma,\gamma') & \mapsto & (\gamma * \gamma' \colon t \mapsto \gamma(t) * \gamma'(t)), \end{array}$$

where Tri is the category of triads of topological spaces, i.e. the category whose object is a triad of a topological space and two base points and whose morphism is a continuous map of underlying topological spaces mapping base points to base points, and where $1 \in G$ is the unit. This multiplication gives a group structure of $\mathrm{Hom}_{\mathrm{Tri}}(([0,1],0,1),(G,1,1))$ and preserve the homotopy equivalence. Therefore it determines a group structure of $\pi_1(G,1)$. It is well-known that those two group structure coincide. Moreover, the group structure of G induce a groupoid structure of $\mathrm{Hom}_{\mathrm{Top}}([0,1],G)$ in a similar way, where Top is the category of topological spaces. The groupoid structure of the fundamental groupoid of G induced by this groupoid structure of $\mathrm{Hom}_{\mathrm{Top}}([0,1],G)$ coincides with the original one. The groupoid structure of $\mathrm{Hom}_{\mathrm{Top}}([0,1],G)$ also induces a (not everywhere defined) binomial on $\mathrm{C}_1(G,\mathbb{Z})$. Since the singular homology $\mathrm{H}_1(G,\mathbb{Z})$ is canonically isomorphic to the Abelian fundamental group $\pi_1(G,1)^{\mathrm{ab}}$, the binomial on $\mathrm{C}_1(G,\mathbb{Z})$ is extended to a group structure on $\mathrm{H}_1(G,\mathbb{Z})$. It is also well-known that the induced group structure of $\mathrm{H}_1(G,\mathbb{Z})$ coincides with the original one. Here we have an analogue. Before that, we introduce two general fucts, which holds even if we do not assume an analytic space is an analytic group.

Lemma 3.2.20. *Let X be a k-$\mathscr{A}$ space, $n \in \mathbb{N}_+$, and $x \in X(k)$. Then the image of the constant morphism*

$$\begin{array}{rcl} x\colon [0,1]^n & \to & X \\ t & \mapsto & x \end{array}$$

in $C^{\square}_n(X)$ is contained in $\ker d_n$*, and* $[x] = 0 \in H^{\square}_n(X)$.

Proof. A constant morphism is obviouly degenerate, and hence $[x] = 0 \in d_{n+1}\mathrm{C}^{\square}_{n+1}(X) \subset \ker d_n \subset \mathrm{C}^{\square}_n(X)$. □

Lemma 3.2.21. *Let X be a k-$\mathscr{A}$ space, and $f \in Hom([0,1],X)$. Set $f^*(t_1) \coloneqq f(1-t_1)$. Then the formal sum of f and f^* in $C_1(X)$ is contained in $d_2C_2(X) \subset \ker d_1$, and hence $[f]+[f^*] = 0 \in H_1(X)$.*

Proof. The non-cubical case: Consider the morphism $F(t_0,t_1,t_2) \equiv f(t_1,t_0+t_2) \in \mathrm{Hom}(\Delta^2,G)$. The one obtains

$$d_2[F] = [f(t_0,t_1)] - [f(0,t_0+t_1)] + [f(t_1,t_0)] = [f] - [f(0,1)] + [f^*]$$

and therefore $[f] + [f^*] \in d_2 C_2^{\triangle}(G)$.

The cubical case: Consider the morphism $F(t_1, t_2) \coloneqq f(t_1(1 - t_2)) \in \mathrm{Hom}([0,1]^2, X)$. Then one obtains

$$d_2[F] = -[f(0)] + [f(1 - t_1)] + [f(t_1)] - [f(0)] = [f] + [f^*] - 2[f(0)]$$

and therefore $[f] + [f^*] \in d_2 C_2^{\square}(X)$. $\square$

Now we consider the singular homology of an analytic group. The group structure of an analytic group G induces a groupoid structure of $\mathrm{Hom}([0,1], G)$ and a group structure of $\mathrm{H}_1(G)$, and the latter one coincides with the original group structure of $\mathrm{H}_1(G)$.

Definition 3.2.22. *A k-$\mathscr{A}$ group is a group object in the category of k-$\mathscr{A}$ spaces.*

Proposition 3.2.23. *Let G be a k-$\mathscr{A}$ group. The sets $Hom(\Delta^n, G)$ and $Hom([0,1]^n, G)$ have a natural group structure induced from the structure morphisms of the group object G. In the case of $n = 1$, this group structure and the evaluation morphisms*

$$\begin{array}{rcl} ev_0, ev_1 \colon Hom([0,1], G) & \rightrightarrows & G(k) \\ f & \mapsto & f(0), f(1) \end{array}$$

induce an Abelian groupoid structure

$$\begin{array}{rcl} \sqcup \colon Hom([0,1], G) \times_k Hom([0,1], G) & \dashrightarrow & Hom([0,1], G) \\ (f, g) \text{ s.t. } f(0) = g(1) & \mapsto & f \sqcup g \coloneqq f * g(1)^{-1} * g, \end{array}$$

where $*$ and $(-)^{-1}$ are the induced multiplication and the induced inverse morphism on $Hom([0,1], G)$. If $H_0(G) = \mathbb{Z}$, then the (not necessarily everywhere defined) multiplication of $H_1(G)$ given by the groupoid structure of $Hom([0,1], G)$ coincides with the multiplication of $H_1(G)$ given by the original group structure of them as the singular homology group, i.e.

$$[f_1] + \cdots + [f_m] = [f_1 \sqcup \cdots \sqcup f_m] \in H_1(G)$$

for any $f_1, \ldots, f_m \in Hom([0,1], G)$ such that $f_1(0) = f_2(1), \ldots, f_{m-1}(0) = f_m(1), f_m(0) = f_1(1) \in G(k)$.

Proof. We omit the cubical case first. To begin with, we see the group structure of $\mathrm{Hom}(\Delta^n, G)$ in duced by the structure morphism of G as a k-$\mathscr{A}$ group.

(multiplication) Two morphisms $f, g \colon \Delta^n \rightrightarrows G$ determine a morphism $f * g \colon \Delta^n \to G$ by the universality of the fibre product $G \times_k G \rightrightarrows G \to k$, Corollary 1.4.20 and Corollary 2.4.20, and the multiplication $G \times_k G \to G$.

(unit) The unit $\mathscr{M}(k) \to G$ and the structure morphism $\Delta^n \to \mathscr{M}(k)$ induces the unit morphism $e \colon \Delta^n \to G$. For any morphism $f \colon \Delta^n \to G$ the products $e * f, f * e \colon \Delta^n \to G$

coincide with f because they are the compositions of f and $G \cong G \times_k k \cong k \times_k G \rightrightarrows G \times_k G \to G$, which are the identity.

(inverse) For any morphism $f \colon \Delta^n \to G$, consider the composition $f^{-1} \colon \Delta^n \to G$ of f and the inverse morphism $G \to G$. Then the products $f * f^{-1}, f^{-1} * f \colon \Delta^n \to G$ coincide with $e \colon \Delta^n \to G$ because they are the composition of f, the diagonal embedding $G \to G \times_k G$, the fibre product $G \times_k G \to G \times_k G$ of the identity and the inverse morphism, and the multiplication $G \times_k G \to G$, where the composition of the latter three morphisms is the unit $G \to G$.

Thus the structure morphisms of G as an analytic group (resp. a dagger group) induce a group structure of $\mathrm{Hom}(\Delta^n, G)$. Let $*$ and $(-)^{-1}$ denote the induced multiplication and the induced inverse map on $\mathrm{Hom}(\Delta^n, G)$.

Now we deal with both of the non-cubical case and the cubical case.

The non-cubical case: Take arbitrary morphisms $f(t_0, t_1), g(t_0, t_1) \in \mathrm{Hom}(\Delta^1, G)$. Consider the morphism $F'(t_0, t_1, t_2) \equiv f(t_0, 1 - t_0) * g(1 - t_2, t_2) \in \mathrm{Hom}(\Delta^2, G)$. The one has

$$\begin{aligned} d_2[F'] &= [f(0,1) * g(1-t_1,t_1)] - [f(t_0, 1-t_0) * g(1-t_1, t_1)] + [f(t_0, 1-t_0) * g(1,0)] \\ &= [f(0,1) * g(t_0,t_1)] - [f(t_0,t_1) * g(t_0,t_1)] + [f(t_0,t_1) * g(1,0)] \end{aligned}$$

and hence $[f * g(1,0)] + [f(0,1) * g] - [f * g] \in d_2 \mathrm{C}_2^{\triangle}(G)$.

The cubical case: Take arbitrary morphisms $f(t_1), g(t_1) \in \mathrm{Hom}([0,1], G)$. Consider the morphism $F'(t_1, t_2) \coloneqq f(t_1 t_2) * g(1)^{-1} * g(t_1) \in \mathrm{Hom}([0,1]^2, G)$. Then one has

$$\begin{aligned} d_2[F'] &= -[f(0) * g(1)^{-1} * g(0)] + [f(t_1) * g(1)^{-1} * g(1)] + [f(0) * g(1)^{-1} * g(t_1)] \\ &\quad -[f(t_1) * g(1)^{-1} * g(t_1)] \\ &= -[f(0) * g(1)^{-1} * g(0)] + [f(t_1)] + [f(0) * g(1)^{-1} * g(t_1)] - [f(t_1) * g(1)^{-1} * g(t_1)] \end{aligned}$$

and hence $[f] + [f(0) * g(1)^{-1} * g] - [f * g(1)^{-1} * g] \in d_2 \mathrm{C}_2^{\square}(G)$.

For any $f, g \in \mathrm{Hom}([0,1], G)$ such that $f(0) = g(1) \in G(k)$, set $f \sqcup g \coloneqq f * g(1)^{-1} * g = f * f(0)^{-1} * g \in \mathrm{Hom}([0,1], G)$. One has $[f] + [g] - [f \sqcup g] \in d_2 \mathrm{C}_2(G)$ by the argument above. Consider any $f, g, h \in \mathrm{Hom}([0,1], G)$ such that $f(0) = g(1), g(0) = h(1) \in G(k)$. Then $(f \sqcup g)(0) = f(0) * g(1)^{-1} * g(0) = g(0) = h(1)$ and $f(0) = g(1) = g(1) * h(1)^{-1} * h(1) = (g \sqcup h)(1)$, and hence both of the pairs $((f \sqcup g), h)$ and $(f, (g \sqcup h))$ are connected. One calculates

$$(f \sqcup g) \sqcup h = (f \sqcup g) * h(1)^{-1} * h = f * g(1)^{-1} * g * h(1)^{-1} * h$$

and

$$f \sqcup (g \sqcup h) = f * (g \sqcup h)(1)^{-1} * (g \sqcup h) = f * g(1)^{-1} * g * h(1)^{-1} * h.$$

It follows that $(f \sqcup g) \sqcup h = f \sqcup (g \sqcup h)$ and $\mathrm{Hom}([0,1], G)$ is an Abelian groupoid.

Finaly for any $f_1, \ldots, f_m \in \mathrm{Hom}([0,1], G)$ such that $f_1(0) = f_2(1), \ldots, f_{m-1}(0) = f_m(1)$, $f_m(0) = f_1(1) \in G(k)$, $[f_1] + \cdots + [f_m], [f_1 \sqcup \cdots \sqcup f_m] \in \ker d_1$ and

$$\begin{aligned}
& [f_1] + \cdots + [f_m] - [f_1 \sqcup \cdots \sqcup f_m] \\
= \ & ([f_1] + [f_2] - [f_1 \sqcup f_2]) + [f_1 \sqcup f_2] + f_3 + \cdots + [f_m] - [f_1 \sqcup f_2 \sqcup \cdots \sqcup f_m] \\
= \ & ([f_1] + [f_2] - [f_1 \sqcup f_2]) + ([f_1 \sqcup f_2] + f_3 - [f_1 \sqcup f_2 \sqcup f_3]) + \cdots \\
& + ([f_1 \sqcup f_2 \sqcup \cdots \sqcup f_{m-1}] + [f_m] - [f_1 \sqcup \cdots \sqcup f_m]) \\
\in \ & d_2 \mathrm{C}_2(G).
\end{aligned}$$

Thus one obtained

$$[f_1] + \cdots + [f_m] = [f_1 \sqcup \cdots \sqcup f_m] \in \mathrm{H}_1(G),$$

which was what we wanted. □

Remark that $f_1 \sqcup \cdots \sqcup f_m \in [\mathrm{Hom}([0,1], G)] \cap \ker d_1$. Indeed, one has

$$f_1 \sqcup \cdots \sqcup f_m = f_1 * f_2(1)^{-1} * f_2 * \cdots * f_m(1)^{-1} * f_m$$

and hence

$$\begin{aligned}
& (f_1 \sqcup \cdots \sqcup f_m)(0) = f_1(0) * f_2(1)^{-1} * f_2(0) * \cdots * f_m(1)^{-1} * f_m(0) = f_m(0) \\
& (f_1 \sqcup \cdots \sqcup f_m)(1) = f_1(1) * f_2(1)^{-1} * f_2(1) * \cdots * f_m(1)^{-1} * f_m(1) = f_1(1) \\
\therefore \ & (f_1 \sqcup \cdots \sqcup f_m)(0) = f_m(0) = f_1(1) = (f_1 \sqcup \cdots \sqcup f_m)(1).
\end{aligned}$$

Corollary 3.2.24. *Let G be an analytically pathwise connected k-$\mathscr{A}$ group. Denote by $Hom(([0,1], 0, 1), (G, 1, 1)) \subset Hom([0,1], G)$ the subset consisting of morphisms $f \colon [0,1] \to G$ such that $f(0) = f(1) = 1 \in G(k)$, where $1 \in G(k)$ is the k-rational point corresponding to the unit morphism $\mathscr{M}(k) \to G$. Note that $Hom(([0,1], 0, 1), (G, 1, 1)) \subset Hom([0,1], G)$ is a subgroup with respect to both the group structure and the groupoid structure of $Hom([0,1], G)$. Then the canonical set-theoretical map*

$$Hom(([0,1], 0, 1), (G, 1, 1)) \twoheadrightarrow [Hom(([0,1], 0, 1), (G, 1, 1))] \hookrightarrow \ker d_1 \twoheadrightarrow H_1(G)$$

is a surjective group homomoprphism with respect to the group strucuture of $Hom(([0,1], 0, 1), (G, 1, 1))$ induced by the groupoid structure of $Hom([0,1], G)$.

Proof. Take any $f, g \in \mathrm{Hom}(([0,1], 0, 1), (G, 1, 1))$. Since $f(0) = 1 = g(1) \in G(k)$, they are connected. One calculates

$$\begin{aligned}
& (f * g)(0) = f(0) * g(0) = 1 = f(1) * g(1) = (f * g)(1) \\
& (f \sqcup g)(0) = g(0) = 1 = f(1) = (f \sqcup g)(1)
\end{aligned}$$

and hence $f * g, f \sqcup g \in \mathrm{Hom}(([0,1], 0, 1), (G, 1, 1))$. It follows that $\mathrm{Hom}(([0,1], 0, 1), (G, 1, 1)) \subset \mathrm{Hom}([0,1], G)$ is a subgroup with respect to both the group structure and the groupoid structure of $\mathrm{Hom}([0,1], G)$.

Take an arbitrary cycle $\xi \in \mathrm{C}_1(G)$. By the difinition of $\mathrm{C}_1(G)$, there exist integers $m, n \in \mathbb{N}$ and $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathrm{Hom}([0,1], G)$ such that $\xi = [f_1] + \cdots + [f_m] - [g_1] - \cdots - [g_n]$. By the argument above, one has

$$\xi \equiv [f_1] + \cdots + [f_m] + [g_1^*] + \cdots + [g_n^*] \pmod{d_2 \mathrm{C}_2(G)},$$

and hence the composition

$$\mathbb{N}^{\oplus \mathrm{Hom}([0,1],G)} \hookrightarrow \mathbb{Z}^{\oplus \mathrm{Hom}([0,1],G)} = \mathrm{Q}_1^{\square}(G) \twoheadrightarrow \mathrm{C}_1(G) \twoheadrightarrow \mathrm{C}_1(G)/d_2\mathrm{C}_2(G)$$

is surjective.

Now take an arbitrary cycle $\overline{\xi} \in \mathrm{H}_1(G)$ and take a representation $\xi \in \ker d_1$. By the argument above, we may and do assume that we have presentations $\xi = [f_1] + \cdots + [f_m]$ by some $m \in \mathbb{N}$ and $f_1, \ldots, f_m \in \mathrm{Hom}([0,1], G)$. If $m = 0$, then obviously

$$\overline{\xi} = 0 = [1] \in \mathrm{im}\Big(\mathrm{Hom}(([0,1],0,1),(G,1,1)) \to \mathrm{H}_1(G)\Big).$$

Assume $m > 0$. Since ξ is a cycle, it follows that

$$0 = d_1\xi = [f_1(0)] - [f_1(1)] + \cdots + [f_m(0)] - [f_m(1)] \in \mathrm{C}_0(G) = \mathbb{Z}^{\oplus G(k)}$$

and therefore arranging the orders of $f_1, \ldots, f_m$, we may assume there exist some $i \in \mathbb{N}_+$ and $0 = a_0 < a_1 < a_2 < \cdots < a_i = m$ such that $f_{a_{j-1}+1}(0) = f_{a_{j-1}+2}(1), \ldots, f_{a_j-1}(0) = f_{a_j}(1)$, $f_{a_j}(0) = f_{a_{j-1}+1}(1)$ for any $j = 1, \ldots, i$. Then one has

$$\begin{aligned} \overline{\xi} &= \overline{[f_1] + \cdots + [f_m]} = \overline{\sum_{j=1}^{i} \sum_{l=a_{j-1}+1}^{a_j} [f_l]} = \overline{\sum_{j=1}^{i} [f_{a_{j-1}+1} \sqcup \cdots \sqcup f_{a_j}]} \\ &= \sum_{j=1}^{i} \overline{[f_{a_{j-1}+1} \sqcup \cdots \sqcup f_{a_j}]} \end{aligned}$$

as elements of $\mathrm{H}_1(G)$. Set $g_j \coloneqq f_{a_{j-1}+1} \sqcup \cdots \sqcup f_{a_j} \in [\mathrm{Hom}([0,1],G)] \cap \ker d_1$ for each $j = 1, \ldots, i$.

Since G is analytically pathwise connected, there exists an analytic path $h_j \colon [0,1] \to G$ such that $h_j(0) = 1, h_j^*(0) = h_j(1) = g_j(0) = g_j(1) \in G(k)$ for each $j = 1, \ldots, i$. One has

$$\overline{[g_j]} = \overline{[g_j]} + \overline{[h_j]} + \overline{[h_j^*]} = \overline{[h_j^*] + [g_j] + [h_j]} = \overline{[h_j^* \sqcup g_j \sqcup h_j]}$$

and

$$\begin{aligned} (h_j^* \sqcup g_j \sqcup h_j)(0) &= h_j(1) * g_j(1)^{-1} * g_j(0) * h_j(1)^{-1} h_j(0) = 1 \\ (h_j^* \sqcup g_j \sqcup h_j)(1) &= h_j(0) * g_j(1)^{-1} * g_j(1) * h_j(1)^{-1} h_j(1) = 1, \end{aligned}$$

which implies $h_j^* \sqcup g_j \sqcup h_j \in \mathrm{Hom}(([0,1],0,1),(G,1,1))$. It follows

$$\begin{aligned} \overline{\xi} &= \sum_{j=1}^{i} \overline{[g_j]} = \sum_{j=1}^{i} \overline{[h_j^* \sqcup g_j \sqcup h_j]} = \overline{[(h_1^* \sqcup g_1 \sqcup h_1) \sqcup \cdots \sqcup (h_i^* \sqcup g_i \sqcup h_i)]} \\ &\in \mathrm{im}\Big(\mathrm{Hom}(([0,1],0,1),(G,1,1)) \to \mathrm{H}_1(G)\Big), \end{aligned}$$

and thus $\mathrm{Hom}(([0,1],0,1),(G,1,1)) \to \mathrm{H}_1(G)$ is surjective. $\square$

3.3 Galois representation $\mathrm{H}_*(X, M)$

In this subsection, we consrtuct the Galois representation $\mathrm{H}^*(X, M) \times G_k \to \mathrm{H}^*(X, M)$. The definition of the Galois actions on the analytic homologies are quite simple because they are just derived from the action on the set of analytic paths.

Definition 3.3.1. *Let X be a k-analytic space, and M a $\mathbb{Z}[G_k]$-module. Define the Galois action on $C^{\triangle}_n(X, M) = \mathbb{Z}^{\oplus Hom(N_k\Delta^n, X)} \otimes_{\mathbb{Z}} M = M^{\oplus Hom(N_k\Delta^n, X)}$ by setting*

$$g\left(\sum_{j=1}^{l}[\gamma_j] \otimes m_j\right) \coloneqq \sum_{j=1}^{m}[\gamma_j \circ g] \otimes g^{-1}(m_j) \in C^{\triangle}_n(X, M)$$

for an element $g \in G_k$, integers $n, m \in \mathbb{N}$, analytic paths $\gamma_1, \ldots, \gamma_m \in\colon N_k\Delta^n \to X$, and elements $m_1, \ldots, m_l \in M$.

Proposition 3.3.2 (Galois representation $\mathrm{H}^{\triangle}_*(X, M)$)**.** *For each $g \in G_k$, the homomorphism $g\colon C^{\triangle}_{\cdot}(X, M) \to C^{\triangle}_{\cdot}(X, M)$ given by the action of $g \in G_k$ is a homomorphism of chain complices. Therefore it induces a Galois representation*

$$\begin{array}{rcl} H^{\triangle}_*(X, M) \times G_k & \to & H^{\triangle}_*(X, M) \\ (\gamma, g) & \mapsto & \gamma \circ g. \end{array}$$

Proof. It suffices to show the equality $(\partial^{(i)}_n \circ \gamma) \circ g = \partial^{(i)}_n \circ (\gamma \circ g)$ for integers $i \leq n \in \mathbb{N}$, an analytic path $\gamma\colon N_k\Delta^n \to X$, and an element $g \in G_k$. Take a representative $\underline{\gamma} \in \mathrm{Hom}(N_k\Delta^n, \mathscr{A}^{\dagger}_k, X)$ of γ. One has

$$\begin{aligned} (\partial^{(i)}_n \circ \underline{\gamma}) \circ g &= (\underline{\gamma}^{(0)}, \partial^{(i)}_n \circ \underline{\gamma}^{(1)}) \circ g = (g^{-1} \circ \underline{\gamma}^{(0)}, \partial^{(i)}_n \circ \underline{\gamma}^{(1)}) \\ &= \partial^{(i)}_n \circ (g^{-1} \circ \underline{\gamma}^{(0)}, \underline{\gamma}^{(1)}) = \partial^{(i)}_n \circ (\underline{\gamma} \circ g) \end{aligned}$$

and hence $(\partial^{(i)}_n \circ \gamma) \circ g = \partial^{(i)}_n \circ (\gamma \circ g)$. □

Definition 3.3.3. *Let X be a k-analytic space, and M a $\mathbb{Z}[G_k]$-module. Define the Galois action on $Q^{\square}_n(X, M) = \mathbb{Z}^{\oplus Hom([0,N_k]^n, X)} \otimes_{\mathbb{Z}} M = M^{\oplus Hom([0,N_k]^n, X)}$ by setting*

$$g\left(\sum_{j=1}^{l}[\gamma_j] \otimes m_j\right) \coloneqq \sum_{j=1}^{m}[\gamma_j \circ g] \otimes g^{-1}(m_j) \in C^{\triangle}_n(X, M)$$

for an element $g \in G_k$, integers $n, m \in \mathbb{N}$, analytic paths $\gamma_1, \ldots, \gamma_m\colon [0, N_k]^n \to X$, and elements $m_1, \ldots, m_l \in M$.

Lemma 3.3.4. *For each $g \in G_k$, the homomorphism $g\colon Q^{\square}_{\cdot}(X, M) \to Q^{\square}_{\cdot}(X, M)$ of sequences of $\mathbb{Z}$-modules given by the action of $g \in G_k$ preserves a degenerate singular cube. Therefore it induces a Galois action on the sequence $C^{\square}_{\cdot}(X, M)$ of $\mathbb{Z}$-modules.*

Proof. For an integer $n \in \mathbb{N}$, take a degenerate analytic path $\gamma \colon [0, N_k]^{n+1} \to X$. By the definition of degeneration, there are integers $i \leq n+1 \in \mathbb{N}_+$ and $\sigma \in \{0, 1\}$ and an analytic path $\gamma_1 \colon [0, N_k]^n \to X$ such that $\gamma = \partial^{i,\sigma}_{n+1} \circ \gamma_1$. For an element $g \in G_k$, one has

$$\gamma \circ g = (\partial^{i,\sigma}_{n+1} \circ \gamma_1) \circ g = \partial^{i,\sigma}_{n+1} \circ (\gamma_1 \circ g)$$

and hence $\gamma \circ g$ is degenerate. It follows $g \colon \mathrm{C}^{\square}_{\cdot}(X, M) \to \mathrm{C}^{\square}_{\cdot}(X, M)$ preserves a degenerate singular cube. $\square$

Proposition 3.3.5 (Galois representation $\mathrm{H}^{\square}_{*}(X, M)$)**.** *For each $g \in G_k$, the homomorphism $g \colon C^{\square}_{\cdot}(X, M) \to C^{\square}_{\cdot}(X, M)$ given by the action of $g \in G_k$ is a homomorphism of chain complices. Therefore it induces a Galois representation*

$$\begin{array}{rcl} H^{\square}_{*}(X, M) \times G_k & \to & H^{\square}_{*}(X, M) \\ (\gamma, g) & \mapsto & \gamma \circ g. \end{array}$$

Proof. The assertion is verified in the totally same way as we did for the analytic singular homology. $\square$

Similarly it is easy to endow the analytic cohomologies with the structure of Galois cohomologies. The well-definedness of the action is verified in the same way as above.

Definition 3.3.6. *Denote by Set the category of sets.*

Definition 3.3.7. *Let X be a k-analytic space, and M a $\mathbb{Z}[G_k]$-module. Define the Galois action on $C^n_{\triangle}(X, M) = Hom_{Grp}(C^{\triangle}_n(X), M) \cong_{\mathbb{Z}} Hom_{Set}(Hom(N_k\Delta^n, X), M)$ by setting*

$$(g \cdot f)(\gamma) \coloneqq g(f(\gamma \circ g))$$

for an element $g \in G_k$, a map $f \colon Hom(N_k\Delta^n, X) \to M$, and an analytic paths $\gamma \in\colon N_k\Delta^n \to X$.

Proposition 3.3.8 (Galois representation $\mathrm{H}^{*}_{\triangle}(X, M)$)**.** *For each $g \in G_k$, the homomorphism $g \colon C^{\cdot}_{\triangle}(X, M) \to C^{\cdot}_{\triangle}(X, M)$ given by the action of $g \in G_k$ is a homomorphism of cochain complices. Therefore it induces a Galois representation*

$$\begin{array}{rcl} H^{*}_{\triangle}(X, M) \times G_k & \to & H^{*}_{\triangle}(X, M) \\ (\gamma, g) & \mapsto & g \cdot \gamma. \end{array}$$

Definition 3.3.9. *Let X be a k-analytic space, and M a $\mathbb{Z}[G_k]$-module. Define the Galois action on $Q^n_{\square}(X, M) = Hom_{Grp}(Q^{\square}_n(X), M) \cong_{\mathbb{Z}} Hom_{Set}(Hom([0, N_k]^n, X), M)$ by setting*

$$(g \cdot f)(\gamma) \coloneqq g(f(\gamma \circ g))$$

for an element $g \in G_k$, a map $f \colon Hom([0.N_k]^n, X) \to M$, and an analytic paths $\gamma \in\colon [0, N_k]^n \to X$.

Lemma 3.3.10. *For each $g \in G_k$, the homomorphism $g\colon Q^{\cdot}_{\square}(X, M) \to Q^{\cdot}_{\square}(X, M)$ of sequences of $\mathbb{Z}$-modules given by the action of $g \in G_k$ preserves a degenerate singular cocube. Therefore it induces a Galois action on the sequence $C^{\cdot}_{\square}(X, M)$ of $\mathbb{Z}$-modules.*

Proposition 3.3.11 (Galois representation $\mathrm{H}^*_{\square}(X, M)$)**.** *For each $g \in G_k$, the homomorphism $g\colon C^{\cdot}_{\square}(X, M) \to C^{\cdot}_{\square}(X, M)$ given by the action of $g \in G_k$ is a homomorphism of cochain complices. Therefore it induces a Galois representation*

$$\begin{array}{rcl} H^*_{\square}(X, M) \times G_k & \to & H^*_{\square}(X, M) \\ (\gamma, g) & \mapsto & g \cdot \gamma. \end{array}$$

4 Integration

Throughout this sction, we consider the case the base field k is a local field, i.e. a complete discrete valuation field with a finite residue field, and we use the rings R, $W(R)$, $\mathrm{B}^+_{\mathrm{dR}}$, and B_{dR}. Brief convension for them is written in §0.1, and we introduce them a little again in the next subsection. We just refer to the definitions and basic facts without proof. For more detail and precise description about them, see [FON].

4.1 Period ring B_{dR}

Definition 4.1.1. *A topological ring is a unital commutative associative ring A endowed with a topology such that the addition $A \times A \to A\colon (a, b) \mapsto a + b$ and the multiplication $A \times A \to A\colon (a, b) \mapsto ab$ are continuous.*

Definition 4.1.2. *Let A be a topological ring. A topological A-module is an A-module M endowed with a topology such that the addition $M \times M \to M\colon (m, n) \mapsto m + n$ and the multiplication $A \times M \to M\colon (a, m) \mapsto am$ are continuous.*

Definition 4.1.3. *Let A be a topological ring. A topological A-algebra is an A-algebra B endowed with a topology such that B is a topological ring and the structure homomorphism $A \to B$ is continuous.*

Definition 4.1.4. *Denote by Mon the category of monoids.*

Definition 4.1.5. *Let φ be Frobenius endomorphism $\overline{k}^{\circ}/p\overline{k}^{\circ} \to \overline{k}^{\circ}/p\overline{k}^{\circ}\colon a \mapsto a^p$. Set*

$$R := \varprojlim_{a \mapsto \varphi(a)} \overline{k}^{\circ}/p\overline{k}^{\circ},$$

and then R is a perfect integral domain over $\mathbb{F}_p$. One has a canonical monoid isomorphism

$$\varprojlim_{a \mapsto a^p} C^{\circ} \to \varprojlim_{a \mapsto \varphi(a)} C^{\circ}/pC^{\circ} \cong \varprojlim_{a \mapsto \varphi(a)} \overline{k}^{\circ}/p\overline{k}^{\circ} = R$$

$$(a_0, a_1, \ldots) \mapsto (a_0 \; mod.p, a_1 \; mod.p, \ldots).$$

Identifying an element of R as that of the multiplicative monoid $\varprojlim C^\circ$*, endow it with the valuation topology given by the multiplicative norm* $|\cdot| \cdot R \to [0,\infty)$ *determined as the composition of the monoid homomorphism*

$$\begin{array}{rcl} \iota\colon R \cong_{(Mon)} \varprojlim_{a \mapsto a^p} C^\circ & \to & C^\circ \hookrightarrow C \\ (a_0, a_1, \ldots) & \mapsto & a_0 \end{array}$$

and the multiplicative norm

$$\begin{array}{rcl} C & \to & [0,\infty) \\ a & \mapsto & |a|. \end{array}$$

This topology is stronger than the project limit topology with respect to the discrete topology of $\overline{k}^\circ / p\overline{k}^\circ$.

Definition 4.1.6. *Set* $R_r := \{a \in R \mid |a| \leq |p|^r\}$ *for each* $r \in [0,\infty)$*. In particular* $R_0 = R$*. The family* $\{R_m \mid m \in \mathbb{N}\}$ *forms a countable basis of neighbourhoods of* $0 \in R$*, and R is a Hausdorff first countable topological ring.*

Lemma 4.1.7. *Fix a system* $\underline{p} = (\underline{p}_0, \underline{p}_1, \ldots) \in \varprojlim C^\circ \cong R$ *of p-power roots of p, i.e.* $\underline{p}_0 = p$ *and* $\underline{p}_{i+1}^p = \underline{p}_i$ *for any* $i \in \mathbb{N}$*. Then one has* $\underline{p}^m R = R_m$*, and hence the topology of R coincides with the* $(\underline{p}R)$*-adic topology.*

Definition 4.1.8. *Since the residue field* $\tilde{k}$ *is perfect, one has the canonical embeddings*

$$\tilde{k} \cong_{\mathbb{F}_p} \varprojlim_{a \mapsto \varphi(a)} \tilde{k} \hookrightarrow \varprojlim_{a \mapsto \varphi(a)} \overline{k}^\circ / p\overline{k}^\circ = R,$$

and regard R as a $\tilde{k}$*-algebra in this way. Since the composition* $\tilde{k} \hookrightarrow R \cong \varinjlim C^\circ \to C^\circ$ *coinsides with Teichmüller embedding* $\tilde{k} \hookrightarrow W(\tilde{k}) \hookrightarrow k^\circ \hookrightarrow C^\circ$*, the relative topology of* $\tilde{k}$ *ub R coincides with the discrete topology.*

Definition 4.1.9. *For a perfect* $\mathbb{F}_p$*-algebra A, denote by* $W(A)$ *the ring of Witt vectors over A. Suppose* $\mathbb{F}_p$ *is endoed with the discrete topology. If A is a topological* $\mathbb{F}_p$*-algebra, endow* $W(A)$ *the direct product topology of A.*

Definition 4.1.10. *Denote by* $k_0 \subset k$ *the absolute unramified subfield* $W(\tilde{k})[p^{-1}] \subset k$*. The embedding* $\tilde{k} \hookrightarrow R$ *induces the canonical embedding*

$$k_0^\circ = W(\tilde{k}) \hookrightarrow W(R),$$

and regard $W(R)$ *as a* k_0°*-algebra in this way.*

Lemma 4.1.11. *The relative topology of k_0° in k coincides with the direct product topology of $W(\tilde{k})$ with respect to the discrete topology of $\tilde{k}$. Endow k_0° with that topology.*

Definition 4.1.12. *Set*

$$W(R_r) := \left\{ \alpha = (\alpha_0, \alpha_1, \dots) \in W(R) \,\middle|\, \alpha_i \in R_r, {}^\forall i \in \mathbb{N} \right\}$$

and

$$W(R_r)_n := W(R_r) + p^n W(R) \subset W(R)$$

for each $r \in [0, \infty)$ and $n \in \mathbb{N}$. The family $\{W(R_m)_n \mid m, n \in \mathbb{N}\}$ forms a basis of neighbourhoods of $0 \in W(R)$, and $W(R)$ is a Hausdorff first countable topological k°-algebra.

Definition 4.1.13. *For a perfect $\mathbb{F}_p$-algebra A, let $[\cdot] \colon A \hookrightarrow W(A)$ be Teichmüller embedding $a \mapsto [a] := (a, 0, \dots)$. Recall that for an Witt vector $\alpha = (\alpha_0, \alpha_1, \dots) \in W(A)$, one has the canonical presentation*

$$\alpha = \sum_{i=0}^{\infty} [\alpha_i^{p^{-i}}] p^i.$$

Lemma 4.1.14.

$$W(R_r)_n \quad = \quad \left\{ \sum_{i=0}^{\infty} [\alpha_i] p^i \,\middle|\, |\alpha_i| \leq |p|^{rp^{-i}}, {}^\forall i = 0, \dots, n-1 \right\}.$$

Corollary 4.1.15.

$$pW(R_r)_n = W(R_{rp})_{n+1} \cap pW(R),$$

and hence the multiplication $W(R) \to W(R) \colon \alpha \mapsto p\alpha$ is a homeomorphism onto the closed image $pW(R) \subset W(R)$.

Lemma 4.1.16. *The embedding $k_0^\circ \hookrightarrow W(R)$ is a homeomorphism onto the image, and $W(R)$ is a topological k_0°-algebra.*

Lemma 4.1.17. *Let A be a topological ring, B a topological A-algebra, and $S \subset B$ a multiplicative set. There is the weakest topology of the localisation B_S in which the canonical A-algebra homomorphim $B \to B_S$ is continuous, and for any continuous A-algebra homomorphim $\phi \colon B \to B'$ satisfying $\phi(S) \subset (B')^\times$, the unique algebraic extension $\phi_S : B_S \to B'$ of ϕ by the universality of the localisation in the category of A-algebras is continuous. In other words, there is the unique localistion B_S possessing the universality of localisation up to isomorphisms in the category of topological A-algebras. Call this topology of B_S the localisation topology with respect to the canonical A-algebra homomorphism $B \to B_S$.*

Proof. A subbase of the localisation topology is given as the collection of the pull-backs of open subsets of topological A-algebra B' by the identity $B_S \to B'$ for any topological A-algebra B' such that the underlying A-algebra of B' is B_S and the canonical A-algebra homomorphism $B \to B_S \to B'$ is continuous. The addition, the multiplication, and the canonical A-algebra homomorpshim $B \to B_S$ are continuous in this topology. □

Definition 4.1.18. *Endow $W(R)[p^{-1}]$ with the localisation topology with respect to the canonical k°-algebra homomorphism $W(R) \hookrightarrow W(R)[p^{-1}]$.*

Definition 4.1.19. *Set*

$$W(R_r)[p^{-1}]_n := p^n W(R) + \sum_{i=0}^{\infty} p^{-i} W(R_{rp^i}) = W(R_r)_n + \sum_{i=1}^{\infty} p^{-i} W(R_{rp^i})$$

for each $r \in [0,\infty)$ and $n \in \mathbb{N}$. The family $\{W(R_m)_n[p^{-1}] \mid m,n \in \mathbb{N}\}$ forms a basis of neighbourhoods of $0 \in W(R)[p^{-1}]$, and $W(R)[p^{-1}]$ is a Hausdorff first countable topological k_0-algebra.

Lemma 4.1.20.

$$W(R_r)[p^{-1}]_n = \left\{ \sum_{i>-\infty}^{\infty} [\alpha_i] p^i \;\middle|\; |\alpha_i| \leq |p|^{rp^{-i}}, {}^\forall i \leq n-1 \right\}.$$

Corollary 4.1.21.

$$W(R_r)[p^{-1}]_n \cap W(R) = W(R_r)_n,$$

and hence the canonical continuous k°-algebra homomorphism $W(R) \hookrightarrow W(R)[p^{-1}]$ is a homeomorphism onto the closed image.

Lemma 4.1.22. *Let A be a topological ring, M an A-module, $(M_i)_{i\in I}$ a family of topological A-modules, and $\phi_i \colon M \to M_i$ a family of A-module homomorphisms. There is the strongest topology of M in topologies such that M is a topological A-module and $\phi_i \colon M \to M_i$ is continuous for any $i \in I$. The same holds for topological A-algebras.*

Proof. A subbase of the strongest topology is given as the collection of the pull-backs of open subsets of M_i by the A-module homomorphisms ϕ_i for each $i \in I$. The addition and the multiplication are continuous in this topology. □

Lemma 4.1.23. *Let A be a topological ring and M be an A-module. There is the unique topology of M such that any A-module homomorphism $M \to N$ to a topological A-module N is continuous. Call it the canonical topology of M.*

Proof. The canonical topology is the strongest topology in topologies such that M is a topological A-module and the identity $M \to N$ is continuous for any topological A-module N whose underlying A-module is M. □

Lemma 4.1.24. *Let A be a topological ring and M be a free A-module of rank $d \in \mathbb{N}$. Any A-module isomorphism $A^d \cong_A M$ is a homeomorphism with respect to the direct product topology of A^d and the canonical topology of M. In particular if A is a Hausdorff first countable topological ring, then M is a Hausdorff first countable topological A-module.*

Lemma 4.1.25. *Let A be a topological ring and B be a finite free A-algebra. Then B is a topological A-algebra with respect to the canonical topology as an A-module.*

Definition 4.1.26. *Set*

$$W(R)_k := W(R) \otimes_{k_0^\circ} k^\circ.$$

Endow $W(R)_k[p^{-1}] = W(R)_k \otimes_{k^\circ} k = W(R)[p^{-1}] \otimes_{k_0} k$ the canonical topology as a finite free $W(R)[p^{-1}]$-module, and regard it a Hausdorff first countable topological $W(R)[p^{-1}]$-algebra.

Definition 4.1.27. *Consider a set-theoretical map*

$$\begin{array}{rcl} \theta\colon W(R) & \to & C^\circ \\ \alpha = \sum_{i=0}^{\infty} [\alpha_i] p^i & \mapsto & \sum_{i=0}^{\infty} \iota(\alpha_i^{p^{-i}}) p^i. \end{array}$$

In fact, θ is a ring homomorphism. Moreover θ is a k_0°-algebra homomorphism because

$$\iota((a, \varphi^{-1}(a), \varphi^{-2}(a), \ldots)) = [a] \in \overline{k}^\circ \subset C^\circ$$

for any element $a \in \tilde{k}$. In particular θ sends $p \in W(R)$ to $p \in C^\circ \backslash \{0\} \subset C^\times$, and hence it has the unique extension $\theta\colon W(R)[p^{-1}] \to C$, which is a k_0-algebra homomorphism. Let $\theta_k\colon W(R)_k[p^{-1}] = W(R)[p^{-1}] \otimes_{k_0} k \to C$ be the unique extension of θ as a k-algebra homomorphism.

Lemma 4.1.28. *Fix a uniformiser $\pi_k \in k$ and a system $\underline{\pi_k} \in R$ of p-power roots of π_k. The kernels of the homomorphisms $\theta_k\colon W(R)_k[p^{-1}] \to C$ and $\theta_k\colon W(R)_k \hookrightarrow W(R)_k[p^{-1}] \to C$ are principal ideals generated by $[\underline{\pi_k}] - \pi_k$, and the $(\ker \theta_k)$-adic topology is Hausdorff, namely*

$$\bigcup_{i=1}^{\infty} (\ker \theta_k)^i = 0 \subset W(R)_k[p^{-1}].$$

In particular replacing k to the closed subfield $k_0 \subset k$, one has the same facts for $\theta\colon W(R) \to C^\circ$ and $\theta\colon W(R)[p^{-1}] \to C$.

Lemma 4.1.29.

$$([\underline{p}] - p)^i W(R) \subset W(R_i)_i$$

and hence $W(R)$ is complete with respect to the $(\ker \theta)$-adic topology.

Definition 4.1.30. *Set*

$$B_{dR}^+ := \varprojlim_{i\to\infty} W(R)_k[p^{-1}]/(\ker\theta_k)^i$$

and endow it the projective limit topology of $W(R)_k[p^{-1}]$. By the previous lemmas, the canonical continuous k°-algebra homomorphim $W(R)_k \to B_{dR}^+$ is injective and its image is closed. Since the composition

$$k \hookrightarrow W(R)_k[p^{-1}] \xrightarrow{\theta_k} C$$

is the identity, the embedding $k \hookrightarrow B_{dR}^+$ is a homeomorphism onto the image.

Definition 4.1.31. *Let $\hat{\theta}_k \colon B_{dR}^+ \to C$ be the canonical extension of $\theta_k \colon W(R)_k[p^{-1}] \to C$. Then $\hat{\theta}_k$ is continuous k-algebra homomorphism and B_{dR}^+ is a complete discrete valuation ring whose maximal ideal is the closed principal ideal $\ker\hat{\theta}_k = ([\underline{\pi_k}] - \pi_k)B_{dR}^+ \subset B_{dR}^+$. In particular B_{dR}^+ is a regular local ring, and hence an integral domain. Set*

$$B_{dR} := Frac B_{dR}^+$$

and endow it the localisation topology with respect to the canonical k-algebra homomorphism $B_{dR}^+ \hookrightarrow B_{dR}$. Since $[\underline{\pi_k}] - \pi_k \in W(R)_k[p^{-1}] \subset B_{dR}^+$ is a generator of the maximal ideal $\ker\hat{\theta}_k \subset B_{dR}^+$, the ring B_{dR} coincides with $B_{dR}^+[([\underline{\pi_k}] - \pi_k)^{-1}]$. Remark that the topology of B_{dR} as a complete discrete valuation field is stronger than the localisation topology.

Definition 4.1.32. *Set*

$$Fil^N B_{dR} := ([\underline{\pi_k}] - \pi_k)^N B_{dR}^+ \subset B_{dR}$$

for each $N \in \mathbb{Z}$. It determines a decreasing filtration $Fil^\cdot$ of B_{dR}^+-submodules of B_{dR}, and is independent of the choices of π_k and $\underline{\pi_k}$ because it coincides with the filtration determined by the discrete valuation.

Lemma 4.1.33. *Since B_{dR}^+ is a complete discrete valuation ring such that it contains k and its residue field is $C/\overline{k}$, there is a canonical way to regard B_{dR}^+ as a $\overline{k}$-algebra compatible with the reduction. Then the embedding $\overline{k} \hookrightarrow B_{dR}^+$ is a homeomorphism onto the image and B_{dR}^+ is a topological $\overline{k}$-algebra.*

Lemma 4.1.34. *The structure of the topological $\overline{k}$-algebras B_{dR}^+ and B_{dR} are independent of the choice of the base field $k/\mathbb{Q}_p$ contained in the fixed algebraically closed field $\overline{k}$ of $\mathbb{Q}_p$.*

By the lemma above, suppose we defined the period ring B_{dR} in the case $k = \mathbb{Q}_p$. Since one does not have to factor through the canonical topology of $W(R)_k[p^{-1}]$, an explicit basis of neighbourhoods of $0 \in \mathrm{B}_{\mathrm{dR}}^+$ is given.

Definition 4.1.35. *Set*

$$U^+_{l,r,n} := W(R_r)[p^{-1}]_n + Fil^l B_{dR} \subset B^+_{dR}$$

for each $r \in [0,\infty)$ and $l,n \in \mathbb{N}$. The family $\{U^+_{l,m,n} \mid l,m,n \in \mathbb{N}\}$ forms a basis of neighbourhoods of $0 \in B^+_{dR}$, and B^+_{dR} is a Hausdorff first countable topological $\overline{k}$-algebra.

Lemma 4.1.36.

$$([\underline{p}] - p)U^+_{l,r,n} = U^+_{l+1,rp,n} \cap Fil^1 B_{dR},$$

and hence the multiplication $B^+_{dR} \to B^+_{dR} \colon \alpha \mapsto ([\underline{p}] - p)\alpha$ is a homeomorphism onto the closed image $Fil^1 B_{dR} \subset B^+_{dR}$.

Lemma 4.1.37. *Set*

$$U_{l,r,n} := \bigcup_{i \in \mathbb{N}} ([\underline{p}] - p)^{-i} U_{l+i,rp^i,n} \subset B_{dR}.$$

for each $r \in [0,\infty)$ and $l,n \in \mathbb{N}$. The family $\{U_{l,m,n} \mid l,m,n \in \mathbb{N}\}$ forms a basis of neighbourhoods of $0 \in B_{dR}$, and B_{dR} is a Hausdorff first countable topological $\overline{k}$-algebra.

Corollary 4.1.38. *The canonical embedding $B^+_{dR} \hookrightarrow B_{dR}$ is a homeomorphism onto the closed image $Fil^0 B_{dR} \subset B_{dR}$.*

We have finished the construction of the topological rings R, $W(R)_k$, $\mathrm{B}^+_{\mathrm{dR}}$, and B_{dR}. They admit continuous Galois actions in the natural way, and the homomorphisms dealt with above are Galois-equivariant.

4.2 Integration of an overconvergent analytic function

We will define the integration of an overconvergent analytic function. We regard a character $x \in \mathbb{Q}^\vee$ as some kind of the exponential map, and therefore we need "log" of a character because the indefinite integral of a function $x = e^{at}$ over the complex plain $\mathbb{C}$ can be presented as $a^{-1}e^{at} + c = (\log x(1))^{-1}x + c$, where log is a suitable branch. In order to define log, we extend the base field k to the field of fraction of a p-adic period ring B_{dR}.

Definition 4.2.1. *Set $\mathbb{Z}[p^{-1}]^\vee := Hom_{Grp}(\mathbb{Z}[p^{-1}], \overline{k}^\times)$. Call an element $x \in \mathbb{Z}[p^{-1}]$ a system of p-power roots of $x(1)$. Define the action of $\mathbb{Z}[p^{-1}]$ on $\mathbb{Z}[p^{-1}]^\vee$ by setting $x^q(t) := x(qt)$ for each $q,t \in \mathbb{Z}[p^{-1}]$ and $x \in \mathbb{Z}[p^{-1}]^\vee$. This action gives $\mathbb{Z}[p^{-1}]^\vee$ a structure of a $\mathbb{Z}[p^{-1}]$-module. Denote by $\mathbb{Z}[p^{-1}]^\vee_+ \subset \mathbb{Z}[p^{-1}]^\vee$ the submonoid of systems of p-power roots of an element of $\overline{k}^\circ$. Then obviously the group completion of $\mathbb{Z}[p^{-1}]^\vee_+$ coincides with $\mathbb{Z}[p^{-1}]^\vee$, because one has $x \in \mathbb{Z}[p^{-1}]^\vee_+$ or $x^{-1} \in \mathbb{Z}[p^{-1}]^\vee_+$ for any $x \in \mathbb{Z}[p^{-1}]^\vee$.*

Definition 4.2.2. *Define a mononid homomorphism $\mathbb{Z}[p^{-1}]^{\vee}_{+} \to R\backslash\{0\}$ as the composition of the monoid homomorphisms*

$$\begin{array}{rcl} \mathbb{Z}[p^{-1}]^{\vee}_{+} & \to & \varprojlim_{z\mapsto z^p} C^{\circ}\backslash\{0\} \\ (x\colon \mathbb{Z}[p^{-1}] \to \overline{k}^{\times}) & \mapsto & (x(1), x(p^{-1}), x(p^{-2}), \ldots) \end{array}$$

and

$$\begin{array}{rcl} \varprojlim_{z\mapsto z^p} C^{\circ} & \to & \varprojlim_{Frob} \overline{k}^{\circ}/p = R \\ a = (a_0, a_1, \ldots) & \mapsto & (a_0 \ mod.p, a_1 \ mod.p, \ldots). \end{array}$$

Then this monoid homomorphism is injective, and regard $\mathbb{Z}[p^{-1}]^{\vee}_{+}$ as a multiplicative submonoid of $R\backslash\{0\}$. By the universality of the group completion, the monoid homomorphism is uniquely extended to a group homomorphism $\mathbb{Z}[p^{-1}]^{\vee} \to Frac(R)^{\times}$, and this is also injective. Regard $\mathbb{Z}[p^{-1}]^{\vee}$ as a multiplicative subgroup of $Frac(R)^{\times}$.

Definition 4.2.3. *For a character $x \in \mathbb{Q}^{\vee}$, consider the restriction $x|_{\mathbb{Z}[p^{-1}]}\colon \mathbb{Z}[p^{-1}] \to \overline{k}^{\times}$. This map $(\cdot)|_{\mathbb{Z}[p^{-1}]}\colon \mathbb{Q}^{\vee} \to \mathbb{Z}[p^{-1}]^{\vee}$ is a $\mathbb{Z}[p^{-1}]$-module homomorphism, and denote by $\mathbb{Q}^{\vee}_{+} \subset \mathbb{Q}^{\vee}$ the preimage of $\mathbb{Z}^{\vee}_{+}$, namely*

$$\mathbb{Q}^{\vee}_{+} = \left\{ x \in \mathbb{Q}^{\vee} \,\middle|\, x(1) \in \overline{k}^{\circ} \right\}.$$

The group completion of $\mathbb{Q}^{\vee}_{+}$ coincides with $\mathbb{Q}^{\vee}$, because one has $x \in \mathbb{Q}^{\vee}_{+}$ or $x^{-1} \in \mathbb{Q}^{\vee}_{+}$ for any $x \in \mathbb{Q}^{\vee}$.

Definition 4.2.4. *Denote by $N_p \subset \mathbb{Q}^{\vee}$ the kernel of the restriction map $(\cdot)|_{\mathbb{Z}[p^{-1}]}\colon \mathbb{Q}^{\vee} \to \mathbb{Z}[p^{-1}]^{\vee}$ and by $T_p \subset \mathbb{Q}^{\vee}$ the $\mathbb{Q}$-linear subspace of characters whose restrictions on $\mathbb{Z}[p^{-1}]$ are torsion elements in $\mathbb{Z}[p^{-1}]^{\vee}$. Then obviously $N_p^{\mathbb{Q}} = T_p$. Set $N_{p,k} \coloneqq N_p \cap \mathbb{Q}^{\vee}_k$ and $T_{p,k} \coloneqq T_p \cap \mathbb{Q}^{\vee}_k$.*

Definition 4.2.5. *For a system $x \in \mathbb{Z}[p^{-1}]^{\vee}$, denote by $[x] \in W(Frac(R))^{\times}$ the Teichmüller embedding $(x, 0, 0, \ldots) \in W(Frac(R))$. If $x \in \mathbb{Z}[p^{-1}]^{\vee}_{+}$, the the image $[x] \in W(Frac(R))^{\times}$ is contained in $W(R)\backslash pW(R) \subset W(Frac(R))^{\times}$. The map*

$$\begin{array}{rcl} [\cdot]\colon \mathbb{Z}[p^{-1}]^{\vee} & \to & W(Frac(R))^{\times} \\ x & \mapsto & [x] \end{array}$$

is a group homomorphism, and the map

$$\begin{array}{rcl} [\cdot]\colon \mathbb{Z}[p^{-1}]^{\vee}_{+} & \to & W(R)\backslash pW(R) \\ x & \mapsto & [x] \end{array}$$

is a monoid homomorphism.

Definition 4.2.6. *Set*

$$[\cdot] := [\cdot] \circ (\cdot)|_{\mathbb{Z}[p^{-1}]} \colon \mathbb{Q}^\vee \xrightarrow{(\cdot)|_{\mathbb{Z}[p^{-1}]}} \mathbb{Z}[p^{-1}]^\vee \xrightarrow{[\cdot]} W(Frac(R))^\times$$
$$\text{and } [\cdot] := [\cdot] \circ (\cdot)|_{\mathbb{Z}[p^{-1}]} \colon \mathbb{Q}^\vee_+ \xrightarrow{(\cdot)|_{\mathbb{Z}[p^{-1}]}} \mathbb{Z}[p^{-1}]^\vee_+ \xrightarrow{[\cdot]} W(R)\backslash pW(R).$$

The first one is a group homomorphism, and the second one is a monoid homomoprhism.

Definition 4.2.7. *Let $x \in \mathbb{Z}[p^{-1}]^\vee$. If $x \in \mathbb{Z}[p^{-1}]^\vee_+$, then $1 - x(-1)[x] \in Fil^1 B_{dR}$. Define the logarithm of x as the element*

$$\log x := -\log\left(1 - \left(1 - \frac{[x]}{x(1)}\right)\right) = \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \frac{[x]}{x(1)}\right)^n \in Fil^1 B_{dR}.$$

This infinite sum converges in $Fil^1 B_{dR}$ because of the $Fil^1 B_{dR}$-adic completeness of B_{dR}.

On the other hand, if $x^{-1} \in \mathbb{Z}^\vee[p^{-1}]_+$, then $1 - x(1)[x^{-1}] \in Fil^1 B_{dR}$. Define the logarithm of x as the element

$$\log x := \log\left(1 - \left(1 - \frac{[x^{-1}]}{x(-1)}\right)\right) = -\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \frac{[x^{-1}]}{x(-1)}\right)^n \in Fil^1 B_{dR}.$$

Note that this definition has the opposite sign compared with the accustomed convension. The reason why we choose this convension is because it will seem to be suitable when we calculate the period of some analytic spaces in §5. We have defined the logarithm of a character $x \in \mathbb{Q}^\vee$ separating it into the two cases, and we have to verify the gluability of the logarithm.

Lemma 4.2.8. *If $x \in \mathbb{Z}^\vee[p^{-1}]_+$ and $x^{-1} \in \mathbb{Z}[p^{-1}]^\vee_+$, one has the equality*

$$-\log\left(1 - \left(1 - \frac{[x]}{x(1)}\right)\right) = \log\left(1 - \left(1 - \frac{[x^{-1}]}{x(-1)}\right)\right),$$

and therefore the map $\log \colon x \mapsto \log x$ determines the well-defined $\mathbb{Z}[p^{-1}]$-linear homomorphism

$$\log \colon \mathbb{Z}[p^{-1}]^\vee \to Fil^1 B_{dR}.$$

Proof. It is the well-known fact of formal power series. □

Definition 4.2.9. *Set*

$$\log := \log \circ (\cdot)|_{\mathbb{Z}[p^{-1}]^\vee} \colon \mathbb{Q}^\vee \xrightarrow{(\cdot)|_{\mathbb{Z}[p^{-1}]^\vee}} \mathbb{Z}[p^{-1}]^\vee \xrightarrow{\log} Fil^1 B_{dR}$$

and call it the logarithm map. The map $\log \colon \mathbb{Q}^\vee \to Fil^1 B_{dR}$ is a $\mathbb{Q}$-linear map.

Moreover, we introduce a reduced logarithm $\log^{\mathrm{r}} \in \mathrm{B}_{\mathrm{dR}}$. A reduced logarithm is not canonically defined, and it depends on the choice of an element $\log p \in \mathrm{B}_{\mathrm{dR}}$.

Definition 4.2.10. *Take an element $l \in B_{dR}$ and a system $\underline{p} \in \mathbb{Q}^\vee$. Since the exact sequences*

$$1 \to \overline{k}^{\circ\times} \to \overline{k}^{\times} \xrightarrow{|\cdot|} |p|^{\mathbb{Q}} \to 1$$

and

$$1 \longrightarrow 1 + \overline{k}^{\circ\circ} \longrightarrow \overline{k}^{\circ\times} \xrightarrow{mod. \overline{k}^{\circ\circ}} \tilde{C}^{\times} \to 1,$$

split, one obtains a logarithm $\log \colon \overline{k} \to B_{dR}$ *setting* $\log(\xi u \underline{p}(q)) = \log u + ql$ *for each* $\xi \in [\tilde{C}]^\times$, $u \in 1 + \overline{k}^{\circ\circ}$, *and* $q \in \mathbb{Q}$. *This logarithm depends on the choice of* $l = \log p \in B_{dR}$, *but is independent of the choice of the system* $\underline{p}$, *which is used to take a splitting of the first exact sequence, because the logarithm of a root of unity is always* 0. *Define* $\log^{r} \colon \mathbb{Q}^\vee \to B_{dR}$ *by* $\log^{r}(x) := \log x - \log x(q_k - 1)$. *The reduced logarithm* $\log^{r}$ *also depends on the choice of* $l = \log p$, *but the restrictions* $\log|_{\overline{k}^{\circ\times}} \colon \overline{k}^{\circ\times} \to \overline{k}$ *and* $\log^{r}|_{\mathbb{Q}_1^\vee} \colon \mathbb{Q}_1^\vee \to \overline{k} + Fil^1 B_{dR}$ *are independent of it, where* $\mathbb{Q}_1^\vee \subset \mathbb{Q}^\vee$ *is the* $\mathbb{Q}$*-vector subspace* $\{x \in \mathbb{Q}^\vee \mid |x(1)| = 1\}$.

The maps log and $\log^{\mathrm{r}}$ are homomorphisms, but not injective. For example, the kernel $N_p \subset \mathbb{Q}^\vee$ of the restriction map $(\cdot)|_{\mathbb{Z}[p^{-1}]} \colon \mathbb{Q}^\vee \to \mathbb{Z}[p^{-1}]^\vee$ is obviously contained in the kernel of log. Conversely, for any element $x \in \mathbb{Q}^\vee$ of the kernel of log, there exists some $n \in \mathbb{N}_+$ such that $x^n|_{\mathbb{Z}[p^{-1}]} = (x|_{\mathbb{Z}[p^{-1}]})^n = 0$ as is shown in the following lemmas:

Lemma 4.2.11. *The homomorphism* $\log \colon \mathbb{Z}[p^{-1}]^\vee \to Fil^1 B_{dR}$ *induces an exact sequence of* $\mathbb{Z}$*-modules*

$$1 \to T \to \mathbb{Z}[p^{-1}]^\vee \to Fil^1 B_{dR},$$

where T *is the subgroup of torsion elements in* $\mathbb{Z}[p^{-1}]^\vee$.

Proof. We have only to prove that $\log x \neq 0$ or $x \in T$ for any $x \in \mathbb{Z}[p^{-1}]^\vee$. We may and do assume $x(1) \in \overline{k}^{\circ}$ without loss of generality. Set $K \colon k(x(1)) \subset \overline{k}$. To begin with, suppose there exists some $l \in \mathbb{N}$ such that $x(p^{-(l+1)}) \notin K$. Take the minimum of such an $l \in \mathbb{N}$. Since $x(p^{-(l+1)}) \notin K$, there exists some $g \in G_K$ such that $(g \cdot x)(p^{-(l+1)}) \neq x(p^{-(l+1)})$. By the definition of $l \in \mathbb{N}$, one has $x(1), x(p^{-1}), \ldots, x(p^{-l}) \in K^\circ$ and this implies

$$\frac{g \cdot x}{x}(p^{-i}) = \frac{(g \cdot x)(p^{-i})}{x(p^{-i})} = \begin{cases} 1 & (i = 0, \ldots, l) \\ \text{a primitive } p\text{-th root of } 1 & (i = l+1) \end{cases}.$$

Therefore one obtains $g \cdot x = \epsilon_{x,g}^{p^l} x$ for some unique system $\epsilon_{x,g} \in \mathbb{Z}[p^{-1}]^\vee$ of p-power roots of unity. Now set $\xi := \log \epsilon_{x,g} = \log(1-(1-[\epsilon_{x,g}])) \in \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$. By the well-known fact, ξ is a generator of the kernel of the canonical homomorphisms $\hat{\theta}_k \colon \mathrm{B}_{\mathrm{dR}}^+ \to C$, and hence $\log x$ can be uniquely presented as $\log x = \sum_{n=1}^\infty [a_{x,n}]\xi^n$ by a unique sequnce $(a_{x,n})_{n \in \mathbb{N}} \in C^{\mathbb{N}}$, where $[\cdot] \colon C \to \mathrm{B}_{\mathrm{dR}}^+$ is the Teichmüller embedding. Set $u := (g \cdot \xi)\xi^{-1} \in \mathrm{B}_{\mathrm{dR}}^+$. Since each of $1 - [g \cdot \epsilon_{x,g}]$ and $1 - [\epsilon_{x,g}]$ is a generator of the kernel of $\hat{\theta}_k$, one has

$$\hat{\theta}_k(u) = \hat{\theta}_k\left(\frac{\log(g \cdot \epsilon_{x,g})}{\log(\epsilon_{x,g})}\right) = \hat{\theta}_k\left(\frac{\sum_{n=1}^\infty n^{-1}\left(1 - [g \cdot \epsilon_{x,g}]\right)^n}{\sum_{n=1}^\infty n^{-1}\left(1 - [\epsilon_{x,g}]\right)^n}\right)$$

$$
\begin{aligned}
&= \hat{\theta}_k\left(\frac{\sum_{n=1}^{\infty} n^{-1}(1-[\epsilon_{x,g}])^{-1}(1-[g\cdot\epsilon_{x,g}])^n}{\sum_{n=0}^{\infty}(n+1)^{-1}(1-[\epsilon_{x,g}])^n}\right) = \frac{\hat{\theta}_k\left(\sum_{n=1}^{\infty} n^{-1}([1-\epsilon_{x,g}])^{-1}(1-[g\cdot\epsilon_{x,g}])^n\right)}{\hat{\theta}_k\left(\sum_{n=0}^{\infty}(n+1)^{-1}(1-[\epsilon_{x,g}])^n\right)} \\
&= \frac{\hat{\theta}_k(([1-\epsilon_{x,g}])^{-1}(1-[g\cdot\epsilon_{x,g}]))}{\hat{\theta}_k(1)} = \hat{\theta}_k\left(\frac{1-[g\cdot\epsilon_{x,g}]}{1-[\epsilon_{x,g}]}\right) \in C^{\circ\times}
\end{aligned}
$$

Now one has

$$
\log(g\cdot x) = g\log x = \sum_{n=1}^{\infty} g[a_{x,n}]u^n\xi^n
$$

and

$$
\log(g\cdot x) - \log x = \log\frac{g\cdot x}{x} = \log\epsilon_{x,g}^{p^l} = p^l\xi
$$

by the continuity of $g\colon \mathrm{B}_{\mathrm{dR}}^+ \to \mathrm{B}_{\mathrm{dR}}^+$. Therefore we conclude

$$
ga_{x,1}\hat{\theta}_k(u) - a_{x,1} = \hat{\theta}_k\left(g[a_{x,1}]u - [a_{x,1}]\right) = \hat{\theta}_k\left(\frac{\log(g\cdot x)-\log x}{\xi}\right) = \hat{\theta}_k\left(p^l\right) = p^l.
$$

This implies $a_{x,1} \notin p^{l+1}C^{\circ}$ and especially $\log x \neq 0$.

Therefore it suffices to show that there exists some $l \in \mathbb{N}$ such that $x(p^{-(l+1)}) \notin K$ for any $x \in \mathbb{Z}[p^{-1}]^{\vee}\backslash T$. Denote by π_K a uniformiser of K and set $x(1) = u\pi_K^j$ by $u \in K^{\circ\times}$ and $j \in \mathbb{N}$. If $j \neq 0$, then obviously the greatest integer $l := [\log_p j] \in \mathbb{Z}$ in the integers not greater than $\log_p j$ satisfies the condition $x(p^{-(l+1)}) \notin K$. On the other hand, suppose $j = 0$, i.e. $x(1) = u \in K^{\circ\times}$. Let $q_K \in \mathbb{N}_+$ be the cardinarity of the residue field $\tilde{K}$ of K°, and then one has $\overline{u}^{q_K} = 1$ in $\tilde{K}$, i.e. $u^{q_K} \in 1 + K^{\circ\circ}$. Replacing x to x^{q_K} and u to u^{q_K}, we may and do assume $u \in 1 + K^{\circ\circ}$ without loss of generality, because the codomain $\mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$ of log is torsion-free. Since $x \notin T$, then there exists some $n \in \mathbb{N}$ such that $x(p^{-n}) \neq 1$. Taking an integer $l \in \mathbb{N}$ such that $x(p^{-n}) \notin 1 + (K^{\circ\circ})^{l+1}$. Set $m := n + [\log_p e_K/(p-1)] + l + 3 \in \mathbb{N}_+$, and assume $x(p^{-m}) \in K$, where $e_K \in \mathbb{N}_+$ is the value of p with respect to the normalised valuation of K and $[\log_p e_K/(p-1)] \in \mathbb{N}$ is the greatest integer in the integers not greater than $\log_p e_K/(p-1)$. Since $x(1) \in 1 + K^{\circ\circ} \subset 1 + \overline{k}^{\circ\circ}$, one obtains $x(p^{-m}) \in K \cap 1 + \overline{k}^{\circ\circ} = 1 + K^{\circ\circ}$. For an integer $s \in \mathbb{N}_+$ and an element $\xi \in 1 + (K^{\circ\circ})^s$, one has

$$
\begin{aligned}
\xi^p &= (1+(\xi-1))^p = 1 + p\sum_{i=1}^{p-1} p^{-1}\begin{pmatrix} p \\ i \end{pmatrix}(\xi-1)^i + (\xi-1)^p \\
&\in 1 + ((K^{\circ\circ})^{ps} + p(K^{\circ\circ})^s) = 1 + (K^{\circ\circ})^{\max\{ps, s+e_K\}} \\
&= 1 + (K^{\circ\circ})^{s+(p-1)\max\{s, e_K/(p-1)\}} \subset 1 + (K^{\circ\circ})^{s+1}
\end{aligned}
$$

and hence

$$
x(p^{-n}) = x(p^{-m})^{p^{m-n}}
$$

$$
\begin{aligned}
&\in (1+(K^{\circ\circ}))^{p^{m-n}} = \left(1+(K^{\circ\circ})^{p^{[\log_p e_K/(p-1)]}}\right)^{p^{m-n-[\log_p e_K/(p-1)]}}\\
&= \left(1+(K^{\circ\circ})^{p^{[\log_p e_K/(p-1)]}}\right)^{p^{l+3}} \subset \left(1+(K^{\circ\circ})^{[e_K/(p-1)]+1}\right)^{p^{l+1}}\\
&= \left(1+(K^{\circ\circ})^{[e_K/(p-1)]+e_K}\right)^{p^l} = 1+(K^{\circ\circ})^{[e_K/(p-1)]+(l+1)e_K} \subset 1+(K^{\circ\circ})^{l+1}.
\end{aligned}
$$

It contradicts the way of the choice of the integer $l \in \mathbb{N}$, and therefore $x(p^{-m}) \notin K$. ◻

Corollary 4.2.12. *The kernel of* $\log\colon \mathbb{Q}^\vee \to \mathit{Fil}^1 B_{dR}$ *is* T_p.

Lemma 4.2.13. *For a character* $x \in \mathbb{Q}^\vee \backslash T_p$*, the logarithm* $\log x$ *is a generator of the principal ideal* $\mathit{Fil}^1 B_{dR} \subset B_{dR}^+$.

Proof. It is well-known that for a system $\epsilon \in \mathbb{Z}[p^{-1}]_+^\vee \subset R$ of primitive p-power roots of unity, the logarithm $\log\epsilon$ is a generator of $\mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$. In particular for a system $\underline{\epsilon} \in \mathbb{Q}^\vee$ of primitive roots of unity the logarithm $\log\underline{\epsilon} = \log(\underline{\epsilon}|_{\mathbb{Z}[p^{-1}]})$ generates $\mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$. Take a character $x \in \mathbb{Q}^\vee\backslash T_p$, and we verify that $\log x$ generates $\mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$. Since $\log x \neq 0$ by the previous corollary and since $\log\epsilon$ generates $\mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$, there are an integer $m \in \mathbb{N}_+$ and an element $u \in \mathrm{Fil}^0\mathrm{B}_{\mathrm{dR}}\backslash\mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}} = (\mathrm{B}_{\mathrm{dR}}^+)^\times$ such that $\log x = u(\log\epsilon)^m$. It is enough to prove $m = 1$. By the argument in the proof of Lemma 4.2.11, one has $x(p^{-i-1}) \notin k(x(1))$ for some integer $i \in \mathbb{N}$, and take the minimum among such integers i. Take ab element $g \in G_{k(x(1))}$ such that $g(x(p^{-i-1})) \neq x(p^{-i-1})$, and set

$$v := \frac{g(\log\epsilon)}{\log\epsilon} = \frac{\log(g(\epsilon))}{\log\epsilon} \in (\mathrm{B}_{\mathrm{dR}}^+)^\times.$$

Since $x(1), \ldots, x(p^{-i}) \in k(x(1))$, the system $y := (g\cdot x)x^{-1} \in \mathbb{Q}^\vee$ satisfies $y(1) = \cdots = y(p^{-i}) = 1$ and $y(p^{-i-1}) \neq 1$. Therefore $y^{p^i} \in \mathbb{Q}^\vee$ is a system of primitive roots of unity, and hence

$$\log y = p^{-i}\log y^{p^i} \in \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}\backslash\mathrm{Fil}^2\mathrm{B}_{\mathrm{dR}}.$$

Then one has

$$
\begin{aligned}
(g(u)v^m - u)(\log\epsilon)^m &= g(u(\log\epsilon)^m) - u(\log\epsilon)^m = g(\log x) - \log x = \log(g\cdot x) - \log x\\
&= \log y \in \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}\backslash\mathrm{Fil}^2\mathrm{B}_{\mathrm{dR}},
\end{aligned}
$$

and this implies $m = 1$. ◻

Now using the logarithm, we formally define the differential and the integral of an analytic function so that $dx/dt_t = (\log x)x$ and $\int x dt_t = x/(\log x)$ for any character $x \in E_{k,1}$.

Definition 4.2.14. *Let* $q_k \geq 2 \in \mathbb{N}$ *be the cardinality* $\#\tilde{k}$ *of the finite residue field* $\tilde{k}$*, as we set in §3.1. Define a k-linear homomorphism* $\int_0^{q_k-1} dt_1\colon k[E_{k,1}] \to B_{dR}$ *setting*

$$\int_0^{q_k-1} f(t_1)dt_1 := \sum_{x\in E_{k,1}\backslash T_{p,k}} f_x\frac{x(q_k-1)-1}{\log x} + (q_k-1)\sum_{x\in T_{p,k}} f_x \in B_{dR}$$

for an element $f = \sum f_x x \in k[E_{k,1}]$.

Definition 4.2.15. *For a sequence $f = (f_x)_{x \in E_{k,1}} \in k^{E_{k,1}}$, we say f is integrable over $[0, q_k - 1]$ if the infinite sum*

$$\int_0^{q_k-1} f(t_1)dt_1 \coloneqq \sum_{x \in E_{k,1}} f_x \int_0^{q_k-1} x(t_1)dt_1$$

converges in B_{dR}. Embedding $k_{[0,q_k-1]} \hookrightarrow k^{E_{k,1}}$, use the term "integrable on $[0, q_k - 1]$" also for an analytic function on $[0, q_k - 1]$.

This functorial has many properties required as an ordinary "integration". Of course an analytic function is not necessarily integrable. We see an example of an integrable function which is not an overconvergent one, and show the compatibility of the involution and the Galois action. After then we will verify the integrability of an overconvergent analytic function.

Definition 4.2.16. *For a character $x \in \mathbb{Q}^\vee$, we say x is a closed character on the closed interval $[0, q_k - 1] \subset \mathbb{R}$ if $x(q_k - 1) = 1$. Denote by $\mathbb{Q}^\vee_{cl(q_k-1)} \subset \mathbb{Q}^\vee$ the $\mathbb{Z}$-submodule of closed characters on $[0, q_k - 1]$. Set $\mathbb{Q}^\vee_{cl(q_k-1),k} \subset \mathbb{Q}^\vee_{cl(q_k-1)} \cap \mathbb{Q}^\vee_k$. For an analytic function $f \in k_{[0,q_k-1]}$, we say f is a closed analytic function on $[0, q_k - 1]$ if $f_x \neq 0$ only when $x \in \mathbb{Q}^\vee_{cl(q_k-1),k}$.*

Lemma 4.2.17.

$$T_{p,k} \subset \mathbb{Q}^\vee_{cl(q_k-1),k}.$$

Proof. Take a character $x \in T_{p,k}$. By the definition of $T_{p,k}$, there exists some $n \in \mathbb{N}_+$ such that $x^n|_{\mathbb{Z}[p^{-1}]} = (x|_{\mathbb{Z}[p^{-1}]})^n = 1 \in \mathbb{Z}[p^{-1}]^\vee$. Since $\mathbb{Z}[p^{-1}]^\vee$ is a $\mathbb{Z}[p^{-1}]$-module, we may and do assune $p \not| n$. Now the element $x(1) \in k$ satisfies $x(1)^n = x^n|_{\mathbb{Z}[p^{-1}]}(1) = 1(1) = 1$, and hence $n \mid (q_k - 1)$. It follows $x(q_k - 1) = 1$. □

Proposition 4.2.18. *A closed analytic function on $[0, q_K - 1]$ is integrable over $[0, q_K - 1]$. In particular, an analytic function presented as a limit of a k-linear combination of elements in T_p is integrable.*

Proof. Take an arbitrary closed analytic function $f \in k_{[0,q_k-1]}$. For a closed character $x \in \mathbb{Q}^\vee_{\mathrm{cl}(q_k-1),k}$, the equality $x(q_k - 1) = 1 = x(0)$ guarantees that $\|x\|_{[0,q_k-1]} = 1$ because the function $|x| \colon [0, q_k - 1] \to [0, \infty) \colon t \mapsto |x(t)| = |x(1)|^t$ is monotonous. One has

$$\int_0^{q_k-1} f(t_1)dt_1 = \sum_{x \in \mathbb{Q}^\vee_k \backslash T_{p,k}} 0 + (q_k - 1) \sum_{x \in T_{p,k}} f_x = (q_k - 1) \sum_{x \in T_{p,k}} f_x$$

and

$$\lim_{x \in T_{p,k}} |f_x| = \lim_{x \in T_{p,k}} |f_x| \|x\|_{[0,q_k-1]} \leq \lim_{x \in \mathbb{Q}^\vee_k} |f_x| \|x\|_{[0,q_k-1]} = 0.$$

This implies the convergence of the integral. □

Lemma 4.2.19. *The integral is invariant under the involution* $*: k_{[0,q_k-1]} \to k_{[0,q_k-1]} \colon f \mapsto f^*$ *defined at Definition 1.1.40. Namely, the involution preserves an integrable analytic function and*

$$\int_0^{q_k-1} f(t_1)dt_1 = \int_0^{q_k-1} f^*(t_1)dt_1$$

for any integrable analytic function $f \in k_{[0,q_k-1]}$. *In particular, the integral of any integrable imaginary analytic function* $f \in \Im k_{[0,q_k-1]}$ *is* 0.

Proof. Trivial because the involution $*: k_{[0,q_k-1]} \to k_{[0,q_k-1]} \colon f \mapsto f^*$ is k-linear and one has

$$x(q_k-1)\frac{x^{-1}(q_k-1)-1}{\log x^{-1}} = \frac{x(q_k-1)-1}{\log x}$$

for any element $x \in E_{k,1} \backslash T_{p,k}$. $\square$

Lemma 4.2.20. *The integral is equivariant under the Galois action* $k_{[0,q_k-1]} \times G_k \to k_{[0,q_k-1]}$ *defined at Definition 1.1.32. Namely, the Galois action preserves an integrable analytic function and*

$$\int_0^{q_k-1} g(f)(t_1)dt_1 = g\left(\int_0^{q_k-1} f(t_1)dt_1\right)$$

for any integrable analytic function $f \in k_{[0,q_k-1]}$.

Proof. Trivial because for any elemenr $g \in G_k$ the action $g \colon k_{[0,q_k-1]} \to k_{[0,q_k-1]} \colon f \mapsto g(f)$ is k-linear and one has

$$\frac{(g\cdot x)(q_k-1)-1}{\log(g\cdot x)} = \frac{g(x(q_k-1))-1}{g\log x} = g\left(\frac{x(q_k-1)-1}{\log x}\right)$$

for any element $x \in E_{k,1} \backslash T_{p,k}$. Note that the $\mathbb{Z}[p^{-1}]$-submodule $T_p \subset \mathbb{Q}^\vee$ is G_k-stable with respect to the Galios representation $\mathbb{Q}^\vee \times G_k \to \mathbb{Q}^\vee$. $\square$

Now it is time to verify the integrability of an overconvergent analytic function. In the proof, we heavily use the topologies of the preriod rings, and hence a reader might recall the definition of them in §4.1 if he or she is not accustomed well.

Theorem 4.2.21 (integrability of an overconvergent analytic function)**.** *An overconvergent analytic function on* $[0, q_k-1]$ *is integrable on* $[0, q_k-1]$.

Proof. Take an overconvergent analytic function $f \in k^\dagger_{[0,q_k-1]}$, and present $f = \sum_{I\in\mathbb{N}^n} f_I x^I = \sum_{I\in\mathbb{N}^n} f_I x_1^{I_1}\cdots x_n^{I_n}$ by an integer $n \in \mathbb{N}$, characters $x = (x_1,\ldots,x_n) \in E^n_{k,1}$, and an overconvergent power series $\sum_{I\in\mathbb{N}^n} f_I T^I \in k\{\|x\|^{-1}_{[0,q_k-1]}T\}^\dagger = k\{\|x_1\|^{-1}_{[0,q_k-1]}T_1,\ldots,\|x_n\|^{-1}_{[0,q_k-1]}T_n\}^\dagger$. Let $r \in \mathbb{N}$ be the greatest integer in the integers such that K has a primitive p^r-th root of

unity. Denote by $F \subset \mathbb{N}^n$ the preimage of $T_{p,k} \subset E_{k,1}$ by the map $\mathbb{N}^n \to E_{k,1} \colon I \mapsto x^I$. For a multi-index $I = (I_1, \ldots, I_n) \in \mathbb{N}^n$, set $x(I) = x^I(1) := x_1(I_1) \cdots x_n(I_n) \in k$ and $I \log x := \log x^I = I_1 \log x_1 + \cdots + I_n \log x_n \in \mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}}$. Denote by $F_{-1}, F_0, F_1 \subset \mathbb{N}^n \backslash F$ the preimages of $k \backslash k^\circ, k^\circ \backslash k^{\circ\circ}, k^{\circ\circ} \subset k$ respectively by the map $\mathbb{N}^n \to k \colon I \mapsto x(I)$. By the denfinition, the integral of f is presented as

$$\begin{aligned}\int_0^{q_k - 1} f(t_1) dt_1 &= \sum_{I \mathbb{N}^n \backslash F} f_I \frac{x((q_k - 1)I) - 1}{I \log x} + (q_k - 1) \sum_{I \in F} f_I \\ &= \left(\sum_{I \in F_1} + \sum_{I \in F_{-1}} + \sum_{I \in F_0} \right) f_I \frac{x((q_k - 1)I) - 1}{I \log x} + (q_k - 1) \sum_{I \in F} f_I.\end{aligned}$$

The final term of the right hand side converges by Proposition 4.2.18, and hence we have only to show that the convergence of the first, second, and third ones respectively. By the continuity of the inclusion $\mathrm{B}_{\mathrm{dR}}^+ \to \mathrm{B}_{\mathrm{dR}}$ and the multiplication $\xi \times (\cdot) \colon \mathrm{B}_{\mathrm{dR}} \to \mathrm{B}_{\mathrm{dR}} \colon a \mapsto \xi a$, it suffices to show that the infinite sum

$$\sum_{I \in F_a} f_I \frac{(x(I) - 1)\xi}{I \log x}$$

converges in $\mathrm{B}_{\mathrm{dR}}^+$ for each $a = -1, 0, 1$. Remark that Lemma 4.2.13. gurantees that $\xi / (I \log x) \in (\mathrm{B}_{\mathrm{dR}}^+)^\times$ for any $x \in \mathbb{Q}^\vee \backslash F$. Take an arbitrary integer $N \in \mathbb{N}$. By the definition of the topology of B_{dR}^+, it suffices to show the convergence of the infinite sum in modulo $\mathrm{Fil}^n \mathrm{B}_{\mathrm{dR}} \subset \mathrm{B}_{\mathrm{dR}}^+$. Fix a uniformiser $\pi_k \in k$ and a system $\underline{\pi_k} \in \mathbb{Z}[p^{-1}]_+^\vee \subset R$ of p-power roots of π_k. Set $\xi := [\underline{\pi_k}] - \pi_k \in W(R)_k \cap \mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}}$. It is well-known that ξ is a generator of the ideals $W(R)_k \cap \mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}} \subset W(R)_k$. Fix a system $\epsilon \in \mathbb{Z}[p^{-1}]_+^\vee \subset R$ of p-power roots of unity. It is also well-known that $[\epsilon^{(p-1)/p}] + [\epsilon^{(p-2)/p}] + \cdots + 1 \in W(R)$ is a generator of the ideal $W(R) \cap \mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}} \subset W(R)$. Let $q \in \mathbb{Q}_+$ be the rational number such that

$$\left| \hat{\theta}_k \left(\frac{\log \epsilon}{\xi} \right) \right| = \left| \theta_k \left(\frac{1 - [\epsilon]}{\xi} \right) \right| = |\pi_k|^q,$$

and set $K := k(\underline{\pi_k}(q)) \subset \overline{k}$. Take a generator $\xi_K \in W(R)_K$ of the principal ideal $W(R)_K \cap \mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}} \subset W(R)_K$. Fix a uniformiser $\pi_K \in K$, and let $e_K \in \mathbb{N}_+$ be the valuation of p in K. Since $\pi_k, \underline{\pi_k}(q) \in K$, one has $e_K / e_k \in \mathbb{N}_+$ and $q e_K / e_k \in \mathbb{N}_+$. Note that $\pi_K^{e_K / e_k} K^\circ = \pi_k K^\circ$ and $\pi_K^{q e_K / e_k} K^\circ = \underline{\pi_k}(q) K^\circ$. Then one has

$$\frac{\log \epsilon}{\xi} \in \pi_K^{q e_K / e_k} W(R)^\times + \sum_{i=1}^{N-1} \frac{\xi_K^i}{i+1} W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}^+.$$

Indeed, set

$$c := \hat{\theta}_k \left(\frac{\log \epsilon}{\xi} \right) \pi_K^{-q e_K / e_k} \in C^{\circ\times}.$$

Since $C^\circ/pC^\circ = \overline{k}^\circ/p\overline{k}^\circ$, there exists a sequence $a = (a_0, a_1, a_2, \ldots) \in (\overline{k}^\circ)^{\mathbb{N}}$ such that $a_0 \in \overline{k}^{\circ\times}$ and $c = a_0 + a_1 p + a_2 p^2 + \cdots$. Take systems $\underline{a_0}, \underline{a_1}, \underline{a_2}, \ldots \in \mathbb{Z}[p^{-1}]_+ \subset R$ of p-power roots of $a_0, a_1, a_2, \ldots$, and set $\alpha \coloneqq [\underline{a_0}] + [\underline{a_1}]p + [\underline{a_2}]p^2 + \cdots \in W(R)^\times$. One has $\theta(\alpha) = c$ by the definition of θ, and hence

$$\begin{aligned}\frac{\log\epsilon}{\xi} - \pi_K^{qe_K/e_k}\alpha &\in \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}} \cap (\underline{\pi_k}(q)W(R)^\times + W(R)_k[(i+1)^{-1}\xi^i \mid i = 1, \ldots, N-1] + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}) \\ &\subset \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}} \cap (W(R)_K[(i+1)^{-1}\xi_K^i \mid i = 1, \ldots, N-1] + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}) \\ &= \sum_{i=1}^{N-1}\frac{\xi_K^i}{i+1}W(R)_K + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}.\end{aligned}$$

It follows that

$$\frac{\log\epsilon}{\xi} \in \pi_K^{qe_K/e_k}W(R)^\times + \sum_{i=1}^{N-1}\frac{\xi_K^i}{i+1}W(R)_K + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}.$$

To begin with, we show that if $x^I(p^{-(l+1)}) \notin k$ for an integer $l \in \mathbb{N}$ and a multi-index $I \in \mathbb{N}^n \backslash F$, then

$$\frac{\xi}{I\log x} \in \sum_{i=0}^{N-1}(N!)^{-i}\pi_K^{-(l+q/e_k)e_K(i+1)}\xi_K^i W(R)_K + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}.$$

Take an element $g \in G_K$ such that $g(x^I(p^{-(l+1)})) \neq x^I(p^{-(l+1)})$. The system $y \coloneqq (g \cdot x^I)x^{-I} \in \mathbb{Q}_k^\vee$ satisfies $y(1) = \cdots = y(p^{-l}) = 1$ and $y(p^{-(l+1)}) \neq 1$, and hence there exists some $u_y \in \mathbb{Z}_p^\times$ such that $y|_{\mathbb{Z}[p^{-1}]} = \epsilon^{u_y p^l}$. It follows that

$$\frac{\log y}{\xi} = \frac{u_y p^l \log\epsilon}{\xi} \in \pi_K^{(l+q/e_k)e_K}W(R)^\times + \pi_K^{le_K}\sum_{i=1}^{N-1}\frac{\xi_K^i}{i+1}W(R)_K + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}},$$

and therefore one has

$$\frac{I\log x}{\xi} \in \left(K^\circ\backslash\pi_K^{(l+q/e_k)e_K+1}K^\circ\right)W(R)_K^\times + \sum_{i=1}^{N-1}\frac{\xi_K^i}{i+1}W(R)_K + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}$$

because the K-algebra homomorphism $\hat{\theta}_K\mathrm{B}_{\mathrm{dR}}^+ \to C$ is G_K-equivariant. This lower bound gives us an upper bound of $\xi/(I\log x)$ in the following calculation:

$$\begin{aligned}\frac{\xi}{I\log x} &\in \left(\left(K^\circ\backslash\pi_K^{(l+q/e_k)e_K+1}K^\circ\right)W(R)_K^\times + \sum_{i=1}^{N-1}\frac{\xi_K^i}{i+1}W(R)_K + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}\right)^{-1} \\ &\subset \left(\left(K^\circ\backslash\pi_K^{(l+q/e_k)e_K+1}K^\circ\right)W(R)_K^\times\left(1 + (N!)^{-1}\pi_K^{-(l+q/e_k)e_K}\xi_K K^\circ W(R)_K + \mathrm{Fil}^N\mathrm{B}_{\mathrm{dR}}\right)\right)^{-1}\end{aligned}$$

$$
\begin{aligned}
&= \pi_K^{-(l+q/e_k)e_K} W(R)_K^\times \left(\sum_{i=0}^{N-1} (N!)^{-i} \pi_K^{-(l+q/e_k)e_K i} \xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}\right) \\
&= \sum_{i=0}^{N-1} (N!)^{-i} \pi_K^{-(l+q/e_k)e_K(i+1)} \xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}.
\end{aligned}
$$

First, for a multi-index $I \in F_1$, present $x(I) = u\pi_k^j \in (k^{\circ\circ})^j \backslash (k^{\circ\circ})^{j+1}$ by a unique element $u \in k^{\circ\times}$ and a unique integer $j \in \mathbb{N}_+$. Since $x_1(1), \ldots, x_n(1) \in k^\times$, there exists some integer $m \in \mathbb{N}_+$ such that $x_1(1), \ldots, x_n(1) \notin p^m k^\circ$. Then obviously one has $x(I) \notin p^{m|I|} k^\circ = \pi_k^{me_k|I|} k^\circ$, i.e. $j < me_k|I|$, where $|I|$ is the sum $I_1 + \cdots + I_n$ of the multi-index $I \in \mathbb{N}^n$ and $e_k \in \mathbb{N}_+$ is the valuation of p in k. Let $l \in \mathbb{N}$ be the minimum among integers such that $x^I(p^{-(l+1)}) \notin k$, and then $l \leq [\log_p j] < [\log_p m|I|]$, where $[\log_p j], [\log_p m|I|] \in \mathbb{N}$ are the greatest integers not greater than $\log_p j$ and $\log_p m|I|$ respectively. Indeed if $l > [\log_p j]$, then one has

$$
x(I) = x^I(p^{-j})^{p^j} \in (k^{\circ\circ})^{p^{[\log_p j]+1}} \subset (k^{\circ\circ})^{j+1}
$$

and it contradicts the definition of the integer j. By the estimation above, one obtains

$$
\begin{aligned}
\frac{\xi}{I \log x} &\in \sum_{i=0}^{N-1} (N!)^{-i} \pi_K^{-(l+q/e_k)e_K(i+1)} \xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}} \\
&\subset \sum_{i=0}^{N-1} (N!)^{-1} \pi_K^{-([\log_p m|I|]+q/e_k)e_K(i+1)} \xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}.
\end{aligned}
$$

Thus the p-adic valuation of the coefficients of lower degree than $N+1$ of $\xi/(I \log x)$ is bounded by the log-scaled upper bound with respect to the multi-index I, and hence the first term of the integral converges because

$$
|f_I(x((q_k-1)I)-1)| = |f_I| = |f_I| \|x^I\|_{[0,q_k-1]} \leq |f_I| \|x\|_{[0,q_k-1]}^I \leq \delta^{|I|}
$$

for some $0 < \delta < 1$ and sufficiently large $I \in F_1$ by the definition of overconvergence and therefore one has

$$
\lim_{I \in F_1} \left| f_I(x((q_k-1)I)-1) \pi_K^{-(\log_p m|I|+q/e_k)e_K(i+1)} \right| \leq |\pi_K|^{-(q/e_k)e_K(i+1)} \delta^{|I|} (m|I|)^{e_K(i+1)\log_p |\pi_K^{-1}|} = 0
$$

for any $i = 0, \ldots, N-1$. It follows that the infinite sum

$$
\sum_{I \in F_1} f_I \frac{(x((q_k-1)I)-1)\xi}{I \log x}
$$

converges in modulo $\mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}$.

Secondly, for a multi-index $I \in F_{-1}$, replacing x to x^{-1}, the same estimation of the p-adic valuations of the coefficients in the ξ-adic development of $\xi/(I \log x) = \xi/(I \log x^{-1})$ as we did for the first term works again. Namely one has

$$\frac{\xi}{I \log x} = -\frac{\xi}{I \log x^{-1}} \in \sum_{i=0}^{N-1} (N!)^{-i} \pi_K^{-([\log_p m|I|]+q/e_k)e_K(i+1)} \xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}$$

and

$$|f_I(x((q_k-1)I)-1)| = |f_I x((q_k-1)I)| = |f_I| \| x^I \|_{[0,q_k-1]} \leq |f_I| \| x \|_{[0,q_k-1]}^I \leq \delta^{|I|}$$

for sufficiently large $I \in F_{-1}$ in the same convension, and the equalities

$$\lim_{I \in F_{-1}} \left| f_I(x((q_k-1)I)-1) \pi_K^{-(\log_p m|I|+q/e_k)e_K(i+1)} \right| \leq |\pi_K|^{-(q/e_k)e_K(i+1)} \delta^{|I|} (m|I|)^{e_K(i+1)\log_p |\pi_K^{-1}|} = 0$$

for integers $i = 0, \ldots, N-1$ gurantee that the infinite sum

$$\sum_{I \in F_{-1}} f_I \frac{(x((q_k-1)I)-1)\xi}{I \log x}$$

converges in modulo $\mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}$.

Thirdly, for a multi-index $I \in F_0$, present $x(I) = \zeta u$ by a unique root $\zeta \in k^{\circ\times}$ of unity of order $h \mid q_k - 1 \in \mathbb{N}_+$ coprime with p and a unique unit $u \in 1 + k^{\circ\circ}$. Note that ζ is the canonical lift of the image $\overline{\zeta} = \overline{x(I)} \in \tilde{k}$ of $x(I)$. There are three cases. One is the case $u \notin 1 + \pi_k p^{r+1} k^\circ$, another one is the case $u \in 1 + \pi_k p^{r+1} k^\circ$ and $h \neq 1$, and the other one is the case $x(I) = u \in 1 + \pi_k p^{r+1} k^\circ$ and $h = 1$. Consider the case $u \notin 1 + \pi_k p^{r+1} k^\circ$. One has

$$x^{hI}(1) = u^h \notin 1 + \pi_k p^{r+1} k^\circ = 1 + \pi_k^{1+(r+1)e_k} k^\circ$$

because $p \nmid h \mid q_K - 1$, and it implies

$$x^{hI}\left(p^{-([\log_p e_K/(p-1)]+(r+1)e_k+3)}\right) \notin k$$

by the argument in the proof of Lemma 4.2.11. Therefore one obtains

$$\frac{\xi}{I \log x} = h \frac{\xi}{hI \log x} \in \sum_{i=0}^{N-1} (N!)^{-i} \pi_K^{-(([\log_p e_K/(p-1)]+(r+1)e_k+3)+q/e_k)e_K(i+1)} \xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}.$$

This implies the existence of an upper bound of the p-adic valuation of each coefficient of lower degree than $N+1$ of the ξ-adic development of $\xi/(I \log x)$ independent of I, and hence the contribution of these terms to the third term of the integral converges because $\lim_{|I| \to \infty} f_I(x((q_k-1)I)-1) = 0$. Consider the next case $u \in 1 + \pi_k p^{r+1} k^\circ$ and $h \neq 1$. Then u has a p^{r+1}-th root in k because the convergence domain of the power series

$$(1+z)^{1/p} := \sum_{i=0}^{\infty} \binom{p^{-1}}{i} z^i$$

$$= \quad 1 + p^{-1}z + \sum_{i=2}^{\infty}\left(\prod_{j=1}^{i-1}(1-pj)\right)p^{-i}z^i \in \mathbb{Q}_p[[z]] \subset C[[z]]$$

is the open ball $\{z \in C \mid |z| < |p|\}$ and it sends the closed ball $\pi_k p^s k^\circ$ to the closed ball $\pi_k p^{s-1}k^\circ$ for any integer $s \in \mathbb{N}_+$. Take a system $\underline{\zeta} \in \mathbb{Z}[p^{-1}]^\vee_+ \subset R$ of p-power roots of ζ satisfying $\underline{\zeta}^h$ is the system of p-power roots of unity. Remark that the latter condition is necessary in order to except the cases $\underline{\zeta}(p^{-1})$ is presented as ζ^a by some integer $a \in \mathbb{N}$ such that $ap \equiv 1$ [mod.h]. One has

$$\log \underline{\zeta} \in \mathbb{Z}_p^\times h^{-1} p^r \xi \subset W(R)_k^\times p^r \xi.$$

Denote by $\underline{u} \in \mathbb{Z}[p^{-1}]^\vee \subset R$ the system of p-power roots of u determined by $\underline{u} := x\underline{\zeta}^{-1}$. Since a p^{r+1}-th root of u and a primitive p^{r+1}-th root of unity are contained in k, one has $\underline{u}(p^{-(r+1)}) \in k$. One has

$$\begin{aligned} \log \underline{u} &= p^{r+1} \log \underline{u}^{p^{-(r+1)}} \in p^{r+1} \sum_{i=1}^{N} \frac{\xi^i}{i} W(R)_k + \mathrm{Fil}^{N+1}\mathrm{B}_{\mathrm{dR}} \\ &\subset \sum_{i=1}^{N} (N!)^{-1} p^{r+1} \xi^i W(R)_k + \mathrm{Fil}^{N+1}\mathrm{B}_{\mathrm{dR}} \end{aligned}$$

It follows

$$\begin{aligned} I \log x &= \log \underline{\zeta} + \log \underline{u} \\ &\in p^r W(R)_k^\times \xi + \sum_{i=1}^{N} (N!)^{-1} p^{r+1} \xi^i W(R)_k + \mathrm{Fil}^{N+1}\mathrm{B}_{\mathrm{dR}} \\ &= p^r \xi \left(W(R)_k^\times + \sum_{i=1}^{N-1} (N!)^{-1} p \xi^i W(R)_k + \mathrm{Fil}^{N}\mathrm{B}_{\mathrm{dR}} \right) \\ \therefore \ \frac{\xi}{I \log x} &\in \left(p^r \xi \left(W(R)_k^\times + \sum_{i=1}^{N-1} (N!)^{-1} p \xi^i W(R)_k + \mathrm{Fil}^{N}\mathrm{B}_{\mathrm{dR}} \right) \right)^{-1} \\ &\subset p^{-r} W(R)_k^\times + \sum_{i=1}^{N-1} (N!)^{-i} p^{-r+i} \xi^i W(R)_k + \mathrm{Fil}^{N}\mathrm{B}_{\mathrm{dR}} \end{aligned}$$

and hence the p-adic valuation of each coefficient has an upper bound independent of I. In conclusion, the contribution of these terms to the third term of the integral converges, too. Now consider the final case $x(I) = u \in 1 + \pi_k p^{r+1} k^\circ$ and $h = 1$. If $u = 1$, then $x(I) - 1 = 0$ and therefore we don't have to take care of this case. We assume $u \neq 1$. Let $j \in \mathbb{N}$ be the minimum among integers satisfying $x(I) = u \notin 1 + \pi_k^{j+1} k^\circ$. Since $x(I) \in 1 + \pi_k p^{r+1} k^\circ$, one has $j \geq 1 + (r+1)e_k$. The argument in the proof of Lemma 4.2.11. gurantees that

$$x^I \left(p^{-\left([\log_p e_k/(p-1)] + j + 3\right)} \right) \notin k,$$

and hence one obtains

$$\begin{aligned}\frac{\xi}{I\log x} &\in \sum_{i=0}^{N-1}(N!)^{-i}\pi_K^{-(([\log_p e_k/(p-1)]+j+3)+q/e_k)e_K(i+1)}\xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}} \\ &\subset \sum_{i=0}^{N-1}(N!)^{-i}\pi_K^{-([\log_p e_k/(p-1)]+(1+(r+1)e_k)+3+q/e_k)e_K(i+1)}\xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}} \\ &\subset \sum_{i=0}^{N-1}(N!)^{-i}\pi_K^{-([\log_p e_k/(p-1)]+(r+1)e_k+4+q/e_k)e_K(i+1)}\xi_K^i W(R)_K + \mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}\end{aligned}$$

Thus the p-adic valuation of each coefficient has an upper bound independent of I, and the contribution of these terms to the third term of the integral converges. We have verified the convergence of all terms in the integral. □

We extend the definition of the integral to the classes of B_{dR}-valued overconvergent analytic functions and overconvergent analytic functions with multi-variables in natural ways.

Definition 4.2.22. *We also call the element of $Bk_S^\dagger := k_S^\dagger \otimes_k B_{dR}$ an overconvergent analytic function on a polytope S.*

Definition 4.2.23 (Galois representation)**.** *The action of G_k on $Bk_S^\dagger$ is given by setting $g(f\otimes b) := g(f)\otimes g(b)$ for each $g\in G_k$, $f\in k_S^\dagger$, and $b\in B_{dR}$.*

Definition 4.2.24. *Extend the integration to a B_{dR}-linear homomorpshim $Bk^\dagger_{[0,q_k-1]} \to B_{dR}$ in the following natural way:*

$$\begin{aligned}\int_0^{q_k-1} dt_1\colon Bk^\dagger_{[0,q_k-1]} &\to B_{dR} \\ \sum_{j=1}^m f_j\otimes b_j &\mapsto \sum_{j=1}^m b_j\int_0^{q_k-1} f_j(t_1)dt_1.\end{aligned}$$

Proposition 4.2.25. *The integration $\int_0^{q_k-1} dt_1\colon Bk^\dagger_{[0,q_k-1]} \to B_{dR}$ is Galois-equivariant.*

Proof. The continuity of the Galois representations $\mathrm{B}_{\mathrm{dR}}\times G_k \to \mathrm{B}_{\mathrm{dR}}$ and $k^\dagger_{[0,q_k-1]}\times G_k \to k^\dagger_{[0,q_k-1]}$ gurantees the assertion because one has

$$\begin{aligned}\int_0^{q_k-1} g(x\otimes b)dt_1 &= \int_0^{q_k-1}(g\cdot x)\otimes g(b)dt_1 = \begin{cases} g(b)\frac{g(x(q_k-1))-1}{\log(g\cdot x)} & (x\notin T_{p,k}) \\ (q_k-1)g(b) & (x\in T_{p,k})\end{cases} \\ &= \begin{cases} g\left(b\frac{x(q_k-1)-1}{\log x}\right) & (x\notin T_{p,k}) \\ g((q_k-1)b) & (x\in T_{p,k})\end{cases} = g\left(\int_0^{q_k-1} x\otimes b dt_1\right)\end{aligned}$$

for each $g\in G_k$, $x\in E_{k,1}$, and $b\in \mathrm{B}_{\mathrm{dR}}$. □

Theorem 4.2.26 (Fubini's theorem). *Let $n \in \mathbb{N}$ be an integer. For an overconvergent analytic function $f \in k^{\dagger}_{[0,q_k-1]^n}$, the infinite sum*

$$\begin{aligned}\int_{[0,q_k-1]^n} f dt_1 \wedge \cdots \wedge dt_n &= \int_0^{q_k-1} \cdots \int_0^{q_k-1} f(t_1, \ldots, t_n) dt_1 \cdots dt_n \\ &:= \sum_{x \in E_{k,n}} f_x \left(\int_0^{q_k-1} x_1(t_1) dt_1 \right) \cdots \left(\int_0^{q_k} x_n(t_1) dt_1 \right)\end{aligned}$$

converges in B_{dR}. Therefore one obtains the G_k-equivariant B_{dR}-linear homomorphism

$$\int_{[0,q_k-1]^n} dt_1 \wedge \cdots \wedge dt_n \colon Bk^{\dagger}_{[0,q_k-1]^n} \to B_{dR}.$$

Proof. Trivial by the isometric isomorphism $k^{\dagger}_{[0,q_k-1]^n} \cong_k (k^{\dagger}_{[0,q_k-1]})^{\otimes^{\dagger} n}$ in Proposition 2.2.11. Just use the obvious convergence $\delta^{|I|}|I|^n \to 0$ in the estimation of the p-adic valuations of the coefficients of the ξ-adic developments of the terms in the infinite sum. The G_k-equivariance is easily seen by Lemma 4.2.20. □

We see the relation between the integration and the differential. The fundamental theorem of calculus is important because it is the most basic implication of Stokes' theorem. We will prove Stokes' theorem in §4.4. Stoke's theorem is necessary to construct the canonical Galois-compatible pairing between the analytic homology and cohomologies in §4.6. To begin with, we define the differential.

Definition 4.2.27. *Define a k-linear homomorphism $d/dt_1 \colon k^{E_{k,1}} \to B_{dR}^{E_{k,1}}$ setting*

$$\frac{d}{dt_1} f(t_1) = \frac{df}{dt_1} := (f_x(\log x)x)_{x \in E_{k,1}} \in B_{dR}^{E_{k,1}}.$$

Lemma 4.2.28. *For a thick polytope $S \subset \mathbb{R}$, embedding $k^{\dagger}_S \hookrightarrow k^{E_{k,1}}$ and $Bk^{\dagger}_S \hookrightarrow k^{E_{k,1}} \otimes_k B_{dR} \hookrightarrow B_{dR}^{E_{k,1}}$, the k-linear homomorphism $d/dt_1 \colon k^{E_{k,1}} \to B_{dR}^{E_{k,1}}$ induces a k-linear homomorphism $d/dt_1 \colon k^{\dagger}_S \to Bk^{\dagger}_{[0,q_k-1]}$.*

Proof. It suffices to show that the image $df/dt_1 \in \mathrm{B}_{\mathrm{dR}}^{E_{k,1}}$ is contained in $\mathrm{B}k^{\dagger}_S$ for any overconvergent analytic function $f \in k^{\dagger}_S$. Present $f = \sum_{I \in \mathbb{N}^n} F_I x^I$ by an integer $n \in \mathbb{N}$, characters $x = (x_1, \ldots, x_n) \in E_{k,1}$, and an overconvergent power series $F = \sum_{I \in \mathbb{N}^n} F_I T^I \in k\{\|x\|_S^{-1} T\}^{\dagger} = k\{\|x_1\|_S^{-1} T_1, \ldots, \|x_n\|_S^{-1} T_n\}^{\dagger}$. One has

$$\begin{aligned}\frac{df}{dt_1} &= \Big(f_y(\log y)\Big)_{y \in E_{k,1}} = \left(\sum_{x^I = y} F_I (I \log x)\right)_{y \in E_{k,1}} \\ &= \sum_{i=1}^{n} \left(\sum_{x^I = y} F_I I_i\right)_{y \in E_{k,1}} \otimes \log x_i = \sum_{i=1}^{n} \frac{\partial F}{\partial T_i}(x) x_i \otimes \log x_i \\ &\in \mathrm{B}k^{\dagger}_S \subset \mathrm{B}_{\mathrm{dR}}^{E_{k,1}}.\end{aligned}$$

□

Lemma 4.2.29. *The differential* $d/dt_1 \colon k_S^\dagger \to Bk_{[0,q_k-1]}^\dagger$ *is* G_k*-equivariant.*

Proof. It is obvious because of the continuity of Galois action on $k_S^\dagger$ and the equality

$$\frac{d(g \cdot x)}{dt_1} = (\log g \cdot x)(g \cdot x) = (g \log x)(g \cdot x) = g((\log x)x)$$

for elements $g \in G_k$ and $x \in E_{k,1}$. □

Definition 4.2.30. *Extend the differential to a* B_{dR}*-linear endomomorphism on* $Bk_{[0,q_k-1]}^\dagger$ *in the following natural way:*

$$\begin{aligned} \frac{d}{dt_1} \colon Bk_{[0,q_k-1]}^\dagger &\to Bk_{[0,q_k-1]}^\dagger \\ \sum_{j=1}^m f_j \otimes b_j &\mapsto \sum_{j=1}^m b_j \frac{df_j}{dt_1}. \end{aligned}$$

Definition 4.2.31. *Extend the bounded k-valued character* $i \colon k_{[0,q_k-1]}^\dagger \to k \colon f \mapsto f(i)$ *associated with* $i_p(i) \in [0, q_k - 1]_k(k)$ *to algebraic* B_{dR}*-valued characters on* $Bk_{[0,q_k-1]}^\dagger$ *for each* $i = 0, \ldots, q_k - 1$ *in the following natural way:*

$$\begin{aligned} i \colon Bk_{[0,q_k-1]}^\dagger &\to B_{dR} \\ \sum_{j=1}^m f_j \otimes b_j &\mapsto \sum_{j=1}^m b_j f_j(i). \end{aligned}$$

Proposition 4.2.32 (fundamental theorem of calculus)**.** *For an overconvergent analytic fucntion* $f \in Bk_{[0,q_k-1]}^\dagger$*, one has the equality*

$$\int_0^{q_k-1} \frac{df}{dt_1} dt_1 = f(q_k - 1) - f(0).$$

Proof. It suffices to show the equality for an overconvergent analytic function $f \in k_{[0,q_k-1]}^\dagger$. Take a presentation $f = \sum_{I \in \mathbb{N}^n} F_I x^I$ by an integer $n \in \mathbb{N}$, characters $x = (x_1, \ldots, x_n) \in E_{k,1}$, and an overconvergent power series

$$F = \sum_{I \in \mathbb{N}^n} F_I T^I \in k\{\|x\|_{[0,q_k-1]}^{-1} T\}^\dagger = k\{\|x_1\|_{[0,q_k-1]}^{-1} T_1, \ldots, \|x_n\|_{[0,q_k-1]}^{-1} T_n\}^\dagger.$$

One has

$$\begin{aligned} \int_0^{q_k-1} \frac{df}{dt_1} dt_1 &= \int_0^{q_k-1} \sum_{i=1}^n \sum_{I \in \mathbb{N}^n} F_I (I_i \log x_i) x^I(t_1) dt_1 \\ &= \sum_{i=1}^n \left(\sum_{x^I \notin T_{p,k}} + \sum_{x^I \in T_{p,k}} \right) \int_0^{q_k-1} F_I (I_i \log x_i) x^I(t_1) dt_1 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x^I \notin T_{p,k}} F_I \frac{(x^I(q_k-1)-1)\sum_{i=1}^n (I_i \log x_i)}{I \log x} + \sum_{x^I \in T_{p,k}} F_I \sum_{i=1}^n (I_i \log x_i) \\
&= \sum_{x^I \notin T_{p,k}} F_I(x^I(q_k-1)-1) + \sum_{x^I \in T_{p,k}} 0 \\
&= \sum_{I \in \mathbb{N}^n} F_I(x^I(q_k-1)-1) \\
&= \sum_{I \in \mathbb{N}^n} F_I x^I(q_k-1) - \sum_{I \in \mathbb{N}^n} F_I \\
&= f(q_k-1) - f(0).
\end{aligned}
$$

□

4.3 Overconvergent differential form on a polytope

We want to define the integral of an overconvergent differential i-form ω on a smooth dagger space X along a cycle γ in the i-th analytic homology $\mathrm{H}_i(X)$ as the "integral" of the "pull-back" $\gamma^*\omega$ of the form to the cube $[0, q_k-1]^i$ or the normalised simplex $(q_k-1)\Delta^i$, namely

$$
\int_\gamma \omega := \begin{cases} \int_{[0,q_k-1]^i} \gamma^*\omega & (\gamma \in \mathrm{H}_i(X) = \mathrm{H}_i^\square(X)) \\ \int_{(q_k-1)\Delta^i} \gamma^*\omega & (\gamma \in \mathrm{H}_i(X) = \mathrm{H}_i^\triangle(X)) \end{cases}.
$$

Therefore in order to justify the integration of an overconvergent differential form on a dagger space along a cycle, it suffices to construct the notion of the "pull-back" of an overconvergent differential form by an analytic path and the "integral" of a differential i-form on the cube $[0, q_k-1]^i$ or the normalised simpkex $(q_k-1)\Delta^i$ in a functorial way. To begin with, we define the module of overconvergent differential forms on a polytope in this subsection, and we deal with the integration in the next subsection.

Definition 4.3.1. *For an integer $n \in \mathbb{N}$, define the partial differential operator $\partial/\partial t_i \colon B_{dR}^{E_{k,n}} \to B_{dR}^{E_{k,n}}$ for each $i = 1, \ldots, n$ setting*

$$
\frac{\partial}{\partial t_i} f = \frac{\partial f}{\partial t_i} := (f_x \log x^{(i)})_{x \in E_{k,n}} \in B_{dR}^{E_{k,n}}
$$

for each $f = (f_x)_{x \in E_{k,n}} \in B_{dR}^{E_{k,n}}$.

Lemma 4.3.2. *For a thick polytope $S \subset \mathbb{R}^n$, embedding $Bk_S^\dagger \hookrightarrow k^{E_{k,n}} \otimes_k B_{dR} \hookrightarrow B_{dR}^{E_{k,n}}$, the partial differential operator $\partial/\partial t_i \colon Bk_S^\dagger \to Bk_S^\dagger$ sends $Bk_S^\dagger$ to $Bk_S^\dagger$, and hence induces a B_{dR}-linear homomorphism $\partial/\partial t_i \colon Bk_S^\dagger \to Bk_S^\dagger$ for each $i = 1, \ldots, n$.*

Proof. It suffices to show that the image of the k-linear subspace $k_S^\dagger \subset \mathrm{B}k_S^\dagger$ by the partial differential operator is contained in $\mathrm{B}k_S^\dagger$. Take an overconvergent analytic function $f \in k_S^\dagger$, and present $f = \sum_{I \in \mathbb{N}^m} F_I x^I$ by an integer $m \in \mathbb{N}$, elements $x = (x_1, \ldots, x_m) \in E_{k,n}^m$, and an overconvergent power series $F = \sum F_I T^I \in k\{\|x\|_S^{-1} T\}^\dagger = k\{\|x_1\|_S^{-1} T_1, \ldots, \|x_m\|_S^{-1} T_m\}^\dagger$. Then one has

$$\begin{aligned}
\frac{\partial f}{\partial t_i} &= (f_y \log y^{(i)})_{y \in E_{k,n}} = \left(\sum_{x^I = y} F_I (I \log x^{(i)}) \right)_{y \in E_{k,n}} \\
&= \left(\sum_{x^I = y} \left(F_I \sum_{j=1}^m I_j \log x_j^{(i)} \right) \right)_{y \in E_{k,n}} = \sum_{j=1}^m \left(\sum_{x^I = y} F_I I_j \right)_{y \in E_{k,n}} \otimes \log x_j^{(i)} \\
&= \sum_{j=1}^m \frac{\partial F}{\partial T_j}(x) x_j \otimes \log x_j^{(i)} \in \mathrm{B}k_S^\dagger
\end{aligned}$$

for each $i = 1, \ldots, n$. ◻

Corollary 4.3.3 (chain rule)**.** *For integers $m, n \in \mathbb{N}$, a thick polytope $S \subset \mathbb{R}^n$, overconvergent analytic functions $f = (f_1, \ldots, f_m) \in k_S^\dagger$, and an overconvergent power series $F \in k\{\|f\|_S^{-1} T\}^\dagger = k\{\|f_1\|_S^{-1} T_1, \ldots, \|f_m\|_S^{-1} T_m\}^\dagger$, one has*

$$\frac{\partial F(f)}{\partial t_i} = \sum_{j=1}^m \frac{\partial F}{\partial T_j}(f) \frac{\partial f_j}{\partial t_i}.$$

Lemma 4.3.4.

$$\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j} = \circ \frac{\partial}{\partial t_j} \circ \frac{\partial}{\partial t_i}.$$

Proof. Trivial. ◻

Now we construct a prototype of the module of overconvergent differential forms on a polytope S. It admits a canonical basis as a finite free $k_S^\dagger$, and it coincides with the module of overconvergent differential forms, which will be defined using it if S is thick. The theory of overconvergent differential forms is pretty simple and easy to handle if S is a thick polytope such as a cube $[0, q_k - 1]^n$, but one must not ignore not thick polytopes because one of the most important polytope in the theory of integration is a normalised simplex $(q_k - 1)\Delta^n$, which is of course not thick.

Definition 4.3.5. *For a polytope $S \subset \mathbb{R}^n$, consider the free $\mathbb{Z}$-module*

$$\bigoplus_{i=1}^n \mathbb{Z} dt_i$$

of rank n generated by the formal basis $dt_1, \dots, dt_n$. Set

$$dl := l_1 dt_1 + \cdots + l_n dt_n \in \bigoplus_{i=1}^{n} \mathbb{Z} dt_i$$

for any $l(t_1, \dots, t_n) = l_0 + l_1 t_1 + \cdots + l_n t_n \in L(S)$, and denote by $dL(S) \subset \mathbb{Z}dt_1 \oplus \cdots \oplus \mathbb{Z}dt_n$ the $\mathbb{Z}$-submodule consisting of elements of the form dl for some $l \in L(S)$.

Definition 4.3.6. *For a polytope $S \subset \mathbb{R}^n$, set*

$$B\Omega^1_{k_S^\dagger} :== Bk_S^\dagger \otimes_{\mathbb{Z}} \left(\bigoplus_{i=1}^{n} \mathbb{Z} dt_i / dL(S) \right) = \bigoplus_{i=1}^{n} Bk_S^\dagger dt_i / \sum_{l \in L(S)} Bk_S^\dagger dl,$$

and call it the module of overconvergent differential forms on S or S_k if one needs to make the base field k clear. In particular if S is thick, then $B\Omega_{k_S^\dagger}$ is a free $Bk_S^\dagger$ of rank n generated by the formal basis $dt_1, \dots, dt_n$.

Definition 4.3.7. *Define the Galois action on $B\Omega^1_{k_S^\dagger}$ by setting*

$$g(f \otimes \omega) := g(f) \otimes \omega$$

for an element $g \in G_k$, an overconvergent analytic function $f \in Bk_S^\dagger$, and an element $\omega \in \bigoplus_{i=1}^{n} \mathbb{Z} dt_i / dL(S)$.

In fact, the module of overconvergent differential forms is a finite free $k_S^\dagger$ module of rank $\dim S$. In order to prove this, we justify the pull-back of an overconvergent differential form by an affine map. The pull-back by an isomorphic affine map determines a $k_S^\dagger$-module isomorphism, and hence it is reduced to the case S is thick.

Definition 4.3.8. *Let $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ be polytopes, and $a \colon S \to T$ an affine map. Take a presentation $(A, b) \in M(m, n; \mathbb{Z}) \times \mathbb{Z}^m$ of a. Consider the $\mathbb{Z}$-module homomorphism*

$$\begin{array}{rccl} a^* \colon & \bigoplus_{j=1}^{m} \mathbb{Z} dt_j & \to & \bigoplus_{i=1}^{n} \mathbb{Z} dt_i \\ & \omega = \sum_{j=1}^{m} \omega^{(j)} dt_j & \mapsto & a^*\omega := \sum_{j=1}^{m} \sum_{i=1}^{n} \omega^{(j)} A_{j,i} dt_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{j,i} \omega^{(j)} \right) dt_i. \end{array}$$

Since a sends S to T, one has $a^*(dl) \in dL(S)$ for any $dl \in d(L(T)) \subset \mathbb{Z}dt_1$ by the definition of $L(S)$ and $L(T)$. Therefore it induces the $\mathbb{Z}$-module homomorphism

$$a^* \colon \bigoplus_{j=1}^{m} \mathbb{Z} dt_j / dL(T) \to \bigoplus_{i=1}^{n} \mathbb{Z} dt_i / dL(S).$$

Regarding $Bk_S^\dagger$ as a $Bk_T^\dagger$-algebra through the bounded k-algebra homomorphism $a^* \colon k_T^\dagger \to k_S^\dagger$, one obtains the $Bk_T^\dagger$-module homomorphism

$$a^* \colon B\Omega^1_{k_T^\dagger} \to B\Omega^1_{k_S^\dagger}$$

$$\sum_{h=1}^{l} f_h \otimes \omega_h \ \mapsto \ \sum_{h=1}^{l} a^*(f_h) \otimes a^*(\omega_h).$$

Lemma 4.3.9. *The $Bk_T^\dagger$-module homomorphism $a^* \colon B\Omega^1_{k_T^\dagger} \to B\Omega^1_{k_S^\dagger}$ is independent of the choice of the presentation (A, b) of a.*

Lemma 4.3.10. *For polytopes S, T, U and affine maps $a \colon S \to T$ and $b \colon T \to U$, one has $(b \circ a)^* = a^* \circ b^* \colon B\Omega^1_{k_U^\dagger} \to B\Omega^1_{k_S^\dagger}$.*

Lemma 4.3.11. *For polytopes S, T and an isomorphic affine map $a \colon S \to T$, the associated $k_T^\dagger$-module homomorphism $a^* \colon B\Omega^1_{k_T^\dagger} \to B\Omega^1_{k_S^\dagger}$ is an isomorphism.*

Proof. These lemmas are verified in the totally same way as the corresponding facts in §1.1. ◻

Corollary 4.3.12. *For a polytope S, the module $B\Omega^1_{k_S^\dagger}$ is a free $k_S^\dagger$-module of rank* $\dim S$.

Proof. Use a thick representative. ◻

Definition 4.3.13. *For a polytope $S \subset \mathbb{R}^n$, set*

$$B\Omega^i_{k_S^\dagger} \coloneqq \bigwedge\nolimits^i_{Bk_S^\dagger} B\Omega^1_{k_S^\dagger}$$

for each $i \in \mathbb{N}$. In particular $B\Omega^0_{k_S^\dagger} = Bk_S^\dagger$ and $B\Omega^i_{k_S^\dagger} = 0$ for any $i >$ $\dim S$. There is a non-canonical $Bk_S^\dagger$-module isomorphism $B\Omega^{\dim S}_{k_S^\dagger} \cong_{Bk_S^\dagger} Bk_S^\dagger$ in general. If S is thick, set $dt_H \coloneqq dt_{H_1} \wedge \cdots \wedge dt_{H_i} \in B\Omega^i_{k_S^\dagger}$ for each $H = (H_1, \ldots, H_i) \in \{1, \ldots, n\}^i$. Then $B\Omega^i_{k_S^\dagger}$ admits the canonical $Bk_S^\dagger$ basis $\{dt_H \mid 1 \leq H_1 < \cdots < H_i \leq n\}$, and hence one has the canonical isomorphism $B\Omega^n_{k_S^\dagger} = Bk_S^\dagger dt_1 \wedge \cdots \wedge dt_n \cong_{Bk_S^\dagger} Bk_S^\dagger$.

Definition 4.3.14. *The Galois representation $B\Omega^1_{k_S^\dagger} \times G_k \to B\Omega^1_{k_S^\dagger}$ induces the Galois action on $B\Omega^i_{k_S^\dagger}$ for each $i \in \mathbb{N}$.*

Be careful about the fact that the module of overconvergent differential forms does not possess the universal module with respect to a derivation to a finite module. Consider the kernel $I \subset k_S^\dagger \otimes_k^\dagger k_S^\dagger$ of the multiplication $k_S^\dagger \otimes_k^\dagger k_S^\dagger \twoheadrightarrow k_S^\dagger \colon a \otimes b \mapsto ab$ and the $k_S^\dagger$-module $I/I^2 = I \otimes_{k_S^\dagger \otimes_k^\dagger k_S^\dagger} k_S^\dagger$. By the well-accustomed argument, it is easily shown that the $k_S^\dagger$-module I/I^2 is topologically generated by elements of the form $df \coloneqq f \otimes 1 - 1 \otimes f \mod I^2$ for an overconvergent analytic function $f \in k_S^\dagger$. Moreover considering the case $S \subset \mathbb{R}^n$ is thick, the element df is calculated as $\sum f_x dx$, and therefore I/I^2 is topologically generated by element of the form dx for an element $x \in E_{k,n}$. Though the family $\{dx \mid x \in E_{k,n}\}$ does not form a topological $k_S^\dagger$-basis, it contains infinitely many linearly independent elements as $\mathbb{Q}^\vee$ is an infinite dimensional $\mathbb{Q}$-linear space. In the

definition of the module $\mathrm{B}\Omega^1_{k_S^\dagger}$, we formally consider the elements $dt_1, \ldots, dt_n$, we added new $\mathrm{B}k_S^\dagger$-linear relations. For example if $n = 1$, the relation is the equality $(\log y)x^{-1}dx = (\log x)y^{-1}dy$ for characters $x, y \in E_{k,1}$. Thus the module $\mathrm{B}\Omega_{k_S^\dagger}$ is much smaller than $I/I^2 \otimes_k \mathrm{B}_{\mathrm{dR}}$, and is a finite free $\mathrm{B}k_S^\dagger$-module.

Once the module of overconvergent differential forms on a polytope gets defined, one obtains the definition of the pull-back of an overconvergent differential form on a dagger space by an analytic path in a natural and quite formal way. Note that since a morphism from a polytope to a dagger space always factors through an affinoid dagger space, one does not have to be bothered with a gluing argument. However, one needs a little preparation.

Definition 4.3.15. *For a thick polytope $S \subset \mathbb{R}^n$, define the differential $d\colon B\Omega^0_{k_S^\dagger} \to B\Omega^1_{k_S^\dagger}$ by setting*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} dt_i \in B\Omega^1_{k_S^\dagger}$$

for any $f \in B\Omega^0_{k_S^\dagger} = Bk_S^\dagger$. The defferential induces the derivation $d^i\colon B\Omega^i_{k_S^\dagger} \to B\Omega^{i+1}_{k_S^\dagger}$ for each $i \in \mathbb{N}$ in a natural way using the canonical basis $\{dt_{h_1} \wedge \cdots \wedge dt_{h_i} \mid 1 \leq h_1 < \cdots < h_i \leq n\}$, and the sequence

$$(B\Omega^{\cdot}_{k_S^\dagger}, d^{\cdot}) \coloneqq \left(0 \to B\Omega^0_{k_S^\dagger} \xrightarrow{d^0} B\Omega^1_{k_S^\dagger} \xrightarrow{d^1} B\Omega^2_{k_S^\dagger} \xrightarrow{d^2} \cdots\right)$$

is a chain complex, i.e. $d^{i+1} \circ d^i = 0$ for any $i \in \mathbb{N}$.

Lemma 4.3.16. *The derivation $d^i\colon B\Omega^i_{k_S^\dagger} \to B\Omega^{i+1}_{k_S^\dagger}$ is G_k-equivariant for any $i \in \mathbb{N}$.*

Proof. Trivial by Lemma 4.2.29. □

Definition 4.3.17. *For polytopes S, T and an integral affine map $a\colon S \to T$, the $Bk_T^\dagger$-module homomorphism $a^*\colon B\Omega^1_{k_T^\dagger} \to B\Omega^1_{k_S^\dagger}$ induces the $Bk_T^\dagger$-module homomorphism $a^*\colon B\Omega^j_{k_T^\dagger} \to B\Omega^j_{k_S^\dagger}$ for each $j \in \mathbb{N}$ by the functoriality of the wedge product.*

Definition 4.3.18. *Denote by $(B_{dR}\text{-}Vec)$ the category of B_{dR}-linear spaces.*

Proposition 4.3.19. *The contravariant functor*

$$\begin{array}{rcl} B\Omega^i_{S_k}\colon (S_k, \tau_S, Cov_S) & \to & (B_{dR}\text{-}Vec) \\ T_k & \rightsquigarrow & H^0(T_k, B\Omega^i_{S_k}) \coloneqq B\Omega^i_{k_T^\dagger} \\ (i\colon T_k \hookrightarrow T'_k) & \rightsquigarrow & (i^*\colon B\Omega^i_{k_{T'}^\dagger} \to B\Omega^i_{k_T^\dagger}) \end{array}$$

is a sheaf. If there is no ambiguity of the base field k, write $B\Omega^i_{S_k} = B\Omega^i_S$.

Proof. It directly follows from Tate's acyclicity, Proposition 2.3.3. □

Proposition 4.3.20. *For thick polytopes $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ and an integral affine map $a \colon S \to T$, the homomorphism $a^* \colon B\Omega^{\cdot}_{k_T^\dagger} \to B\Omega^{\cdot}_{k_S^\dagger}$ of sequences of $Bk_T^\dagger$-modules is a homomorphism of chain complices, i.e. $a^* \circ d_j = d_j \circ a^* \colon B\Omega^j_{k_T^\dagger} \to B\Omega^{j+1}_{k_S^\dagger}$ for any $j \in \mathbb{N}$.*

Proof. Take an integer $j \leq m \in \mathbb{N}$. It suffices to show that

$$(d^j \circ a^*)(f dt_J) = (a^* \circ d^j)(f dt_J)$$

for an overconvergent analytic function $f \in k_T^\dagger$ and an increasing sequence $J = (J_1, \ldots, J_i)$ satisfying $1 \leq J_1 < \cdots < J_i \leq m$. Take a presentation $f = \sum_{I \in \mathbb{N}^l} F_I x^I$ by an integer $l \in \mathbb{N}$, parametres $x = (x_1, \ldots, x_l) \in E^l_{k,m}$, and an overconvergent power series $F = \sum_{I \in \mathbb{N}^l} F_I T^I \in k\{\|x\|_T^{-1} T\}^\dagger = k\{\|x_1\|_T^{-1} T_1, \ldots, \|x_l\|_T^{-1} T_l\}^\dagger$, and take a presentation $(A, b) \in \mathrm{M}(m, n; \mathbb{Z}) \times \mathbb{Z}^m$. One has

$$\begin{aligned}
& a^*(f dt_J) = a^*(f)(a^* dt_{J_1}) \wedge \cdots \wedge (a^* dt_{J_i}) \\
= & \sum_{I \in \mathbb{N}^l} F_I \left(\prod_{j'=1}^m x^{(j')}(b_{j'}) \prod_{h=1}^n x^{(j')A_{j',h} I}(t_h) \right) \left(\sum_{h=1}^n A_{J_1,h} dt_h \right) \wedge \cdots \wedge \left(\sum_{h=1}^n A_{J_i,h} dt_h \right) \\
= & \sum_{h_1,\ldots,h_i=1}^n \sum_{I \in \mathbb{N}^l} \left(F_I \prod_{i'=1}^i A_{J_{i'},h_{i'}} \prod_{j'=1}^m x^{(j')}(b_{j'}) \right) \prod_{h=1}^n \left(\prod_{j'=1}^m x^{(j')A_{j',h} I} \right)(t_h) dt_{(h_1,\ldots,h_i)},
\end{aligned}$$

$$\begin{aligned}
& (d^j \circ a^*)(f dt_J) \\
= & \left(\sum_{h_1,\ldots,h_i=1}^n \sum_{I \in \mathbb{N}^l} \left(F_I \prod_{i'=1}^i A_{J_{i'},h_{i'}} \prod_{j'=1}^m x^{(j')}(b_{j'}) \right) \prod_{h=1}^n \left(\prod_{j'=1}^m x^{(j')A_{j',h} I} \right)(t_h) \right) \\
& \qquad \left(\sum_{h_0=1}^n \sum_{J_0=1}^m \log x^{(J_0)A_{J_0,h_0} I} dt_{h_0} \right) \wedge dt_{(h_1,\ldots,h_i)} \\
= & \sum_{J_0=1}^m \sum_{h_0,\ldots,h_i=1}^n \sum_{I \in \mathbb{N}^l} \left(F_I I \log x^{(J_0)} \prod_{i'=0}^i A_{J_{i'},h_{i'}} \prod_{j'=1}^m x^{(j')}(b_{j'}) \right) \prod_{h=1}^n \left(\prod_{j'=1}^m x^{(j')A_{j',h} I} \right)(t_h) dt_{(h_0,\ldots,h_i)},
\end{aligned}$$

$$\begin{aligned}
& d^j(f dt_J) = \sum_{J_0=1}^m \frac{\partial f}{\partial t_{J_0}} dt_{J_0} \wedge dt_J = \sum_{J_0=1}^m \frac{\partial F(x)}{\partial t_{J_0}} dt_{(J_0,\ldots,J_l)} = \sum_{J_0=1}^m \sum_{I \in \mathbb{N}^l} F_I \frac{\partial x^I}{\partial t_{J_0}} dt_{(J_0,\ldots,J_l)} \\
= & \sum_{J_0=1}^m \sum_{I \in \mathbb{N}^l} (F_I I \log x^{(J_0)}) x^I dt_{(J_0,\ldots,J_l)},
\end{aligned}$$

and

$$(a^* \circ d^j)(f dt_J) = a^* \left(\sum_{J_0=1}^m \sum_{I \in \mathbb{N}^l} (F_I I \log x^{(J_0)}) x^I dt_{(J_0,\ldots,J_l)} \right)$$

$$
\begin{aligned}
&= \sum_{J_0=1}^{m} \sum_{I\in\mathbb{N}^l} (F_I I \log x^{(J_0)}) \left(\prod_{j'=1}^{m} x^{(j')}(b_{j'}) \prod_{h=1}^{n} x^{(j')A_{j',h}I}(t_h) \right) \left(\sum_{h=1}^{n} A_{J_0,h} dt_h \right) \wedge \cdots \wedge \left(\sum_{h=1}^{n} A_{J_i,h} dt_h \right) \\
&= \sum_{J_0=1}^{m} \sum_{h_0,\ldots,h_l=1}^{n} \sum_{I\in\mathbb{N}^l} \left(F_I I \log x^{(J_0)} \prod_{i'=0}^{l} A_{J_{i'},h_{i'}} \prod_{j'=1}^{m} x^{(j')}(b_{j'}) \right) \left(\prod_{h=1}^{n} \prod_{j'=1}^{m} x^{(j')A_{j',h}I}(t_h) \right) dt_{(h_0,\ldots,h_i)} \\
&= (d^j \circ a^*)(f dt_J),
\end{aligned}
$$

which was what we wanted. $\square$

Corollary 4.3.21. *Let S and T be thick polytopes, and $a\colon S \to T$ an isomorphic affine map. Then one has $a^* \circ d^i \circ (a^{-1})^* = d^i\colon B\Omega^i_{k_S^\dagger} \to B\Omega^{i+1}_{k_S^\dagger}$ for any $i \in \mathbb{N}$.*

Corollary 4.3.22. *Let S be a polytope. Take a thick polytope T and an isomorphic affine map $a\colon S \to T$. Set*

$$
d^i := a^* \circ d^i \circ (a^{-1})^* \colon B\Omega^i_{k_S^\dagger} \xrightarrow{(a^{-1})^*} B\Omega^i_{k_T^\dagger} \xrightarrow{d^i} B\Omega^{i+1}_{k_T^\dagger} \xrightarrow{a^*} B\Omega^{i+1}_{k_S^\dagger}
$$

for each $i \in \mathbb{N}$. Then the derivations $d^\cdot$ are independent of the choice of T and a, and the sequence

$$
\left(B\Omega^\cdot_{k_S^\dagger}, d^\cdot \right) := \left(0 \to B\Omega^0_{k_S^\dagger} \xrightarrow{d^0} B\Omega^1_{k_S^\dagger} \xrightarrow{d^1} B\Omega^2_{k_S^\dagger} \xrightarrow{d^2} \cdots \right)
$$

is a chain complex, and coincides with the original chain complex we have already define if S is thick.

Definition 4.3.23. *For a polytope S, define the differential $d\colon Bk_S^\dagger \to B\Omega^1_{k_S^\dagger}$ as the composition of the identity $Bk_S^\dagger = B\Omega^0_{k_S^\dagger}$ and the derivation $d^0\colon B\Omega^0_{k_S^\dagger} \to B\Omega^1_{k_S^\dagger}$.*

Lemma 4.3.24. *The differential $d\colon Bk_S^\dagger \to B\Omega^1_{k_S^\dagger}$ satisfies Leibniz rule, and coincides with the original differential we have already define if S is thick.*

Proof. The second assertion is trivial, and we verify Leibniz rule in the case $S \subset \mathbb{R}^n$ is thick. Take overconvergent analytic functions $f, g \in k_S^\dagger$, and present $f = \sum_{I\in\mathbb{N}^m} F_I x^I$ and $g = \sum_{I\in\mathbb{N}^m} G_I x^I$ by an integer $m \in \mathbb{N}$, elements $x = (x_1, \ldots, x_m) \in E^m_{k,n}$, and overconvergent power series

$$
F = \sum F_I T^I, G = \sum_{I\in\mathbb{N}^m} G_I T^I \in k\{\|x\|_S^{-1} T\}^\dagger = k\{\|x_1\|_S^{-1} T_1, \ldots, \|x_m\|_S^{-1} T_m\}^\dagger.
$$

Using Leibniz rule of the formal differential of formal power series, Lemma 2.1.3, one has

$$
fg = F(x)G(x) = FG(x) = \sum_{I\in\mathbb{N}} \left(\sum_{J\leq J\in\mathbb{N}^m} F_{I-J} G_J \right) x^I,
$$

$$\begin{aligned}
d(fg) &= d(FG(x)) = \sum_{j=1}^{n} \frac{\partial FG(x)}{\partial t_j} dt_j = \sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial FG}{\partial T_l}(x) \frac{\partial x_l}{\partial t_j} dt_j \\
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \left(\frac{\partial F}{\partial T_l} G + F \frac{\partial G}{\partial T_l} \right)(x) \frac{\partial x_l}{\partial t_j} dt_j \\
&= \sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial F}{\partial T_l}(x) G(x) \frac{\partial x_l}{\partial t_j} dt_j + \sum_{j=1}^{n} \sum_{l=1}^{m} F(x) \frac{\partial G}{\partial T_l}(x) \frac{\partial x_l}{\partial t_j} dt_j \\
&= G(x) \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial F}{\partial T_l}(x) \frac{\partial x_l}{\partial t_j} dt_j \right) + F(x) \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial G}{\partial T_l}(x) \frac{\partial x_l}{\partial t_j} dt_j \right) \\
&= G(x) \left(\sum_{j=1}^{n} \frac{\partial F(x)}{\partial t_j} dt_j \right) + F(x) \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial G(x)}{\partial t_j} dt_j \right) \\
&= f dg + g df.
\end{aligned}$$

□

Definition 4.3.25. *Let S be a polytope. Denote by $I_S \subset k^{\dagger}_{S\times S}$ the kernel of the bounded k-algebra homomorphism $\mu\colon k^{\dagger}_{S\times S} \to k^{\dagger}_S$ associated with the diagonal map $S \hookrightarrow S \times S$, which is an integral affine map. The $k^{\dagger}_{S\times S}$-module structure of the ideal $I_S \subset k^{\dagger}_{S\times S}$ and the canonical bounded k-algebra isomorphism $k^{\dagger}_{S\times S}/I_S \to k^{\dagger}_S$ induces the $k^{\dagger}_S$-module structure of I_S/I_S^2.*

Definition 4.3.26. *Let $S \subset \mathbb{R}^n$ be a thick polytope. Define a k-linear homomorphism $\iota\colon k^{\dagger}_{S\times S} \to B\Omega^1_{k^{\dagger}_S}$ as the composition of the k-linear homomorphism*

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\partial}{\partial t_i} dt_i \colon k^{\dagger}_{S\times S} &\to \bigoplus_{i=1}^{n} Bk^{\dagger}_{S\times S} dt_i \subset B\Omega^1_{k^{\dagger}_{S\times S}} \\
f &\mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial t_i} dt_i
\end{aligned}$$

and the B_{dR}-linear homomorphism

$$\begin{aligned}
(id \otimes \mu)^n \colon \bigoplus_{i=1}^{n} Bk^{\dagger}_{S\times S} dt_i &\to \bigoplus_{i=1}^{n} Bk^{\dagger}_S dt_i = B\Omega^1_{k^{\dagger}_S} \\
\sum_{i=1}^{n} f_i dt_i &\mapsto \sum_{i=1}^{n} (id \otimes \mu)(f_i) dt_i,
\end{aligned}$$

where $id \otimes \mu\colon Bk^{\dagger}_{S\times S} \to Bk^{\dagger}_S$ is the B_{dR}-algebra homomorphism induced by the bounded k-algebra homomorphism $\mu\colon k^{\dagger}_{S\times S} \to k^{\dagger}_S$.

Lemma 4.3.27. *Let $S \subset \mathbb{R}^n$ be a thick polytope. The kernel of the k-linear homomorphism $\iota\colon k^\dagger_{S\times S} \to B\Omega^1_{k^\dagger_S}$ contains the ideal $I_S^2 \subset k^\dagger_{S\times S}$, and hence it induces a k-linear homomorphism $\iota\colon I_S/I_S^2 \to B\Omega^1_{k^\dagger_S}$.*

Proof. Take overconvergent analytic functions $f, g \in I_S$, and it suffices to show that $\iota(fg) = 0$. Indeed, one has

$$\begin{aligned}\iota(fg) &= (\mathrm{id}\otimes\mu)^n\left(\sum_{i=1}^n \left(f\frac{\partial g}{\partial t_i} + \frac{\partial f}{\partial t_i}g\right)dt_i\right)\\ &= \sum_{i=1}^n\sum_{j=1}^m\left(\mu(f)(\mathrm{id}\otimes\mu)\left(\frac{\partial g}{\partial t_i}\right) + (\mathrm{id}\otimes\mu)\left(\frac{\partial f}{\partial t_i}\right)\mu(g)\right)dt_i = 0.\end{aligned}$$

□

Lemma 4.3.28. *Let S and T be thick polytopes, and $a\colon S \to T$ an integral affine map. Then the direct product $a\times a\colon S\times S \to T\times T$ is also an integral affine map, and the diagram*

$$\begin{array}{ccc} k^\dagger_{T\times T} & \xrightarrow{\iota} & B\Omega^1_{k^\dagger_T} \\ {\scriptstyle (a\times a)^*}\downarrow & & \downarrow{\scriptstyle a^*} \\ k^\dagger_{S\times S} & \xrightarrow{\iota} & B\Omega^1_{k^\dagger_S} \end{array}$$

commutes.

Proof. Then the direct product $a\times a\colon S\times S \to T\times T$ is presented by the direct sum of two copies of the presentation of a. The bounded k-algebra homomorphism μ coincides with the multiplication $k^\dagger_S \otimes^\dagger k^\dagger_S \to k^\dagger_S$ identifying $k^\dagger_{S\times S} = k^\dagger_S \otimes^\dagger_k k^\dagger_S$ by the canonical isometric isomorphism in Proposition 2.2.11. Both of the multiplication and the partial differentials commute with a^*, and hence we have done. □

Corollary 4.3.29. *Let S and T be thick polytopes, and $a\colon S \to T$ an isomorphic integral affine map. Then one has*

$$a^*\circ\iota\circ(a^{-1}\times a^{-1})^* = \iota\colon k^\dagger_{S\times S} \to B\Omega^1_{k^\dagger_S}.$$

Corollary 4.3.30. *Let S be a polytope. Take a thick polytope T and an isomorphic integral affine map $a\colon S \to T$. Set*

$$\iota \coloneqq a^*\circ\iota\circ(a^{-1}\times a^{-1})^*\colon k^\dagger_{S\times S} \xrightarrow{(a^{-1}\times a^{-1})^*} k^\dagger_{T\times T} \xrightarrow{\iota} B\Omega^1_{k^\dagger_T} \xrightarrow{a^*} B\Omega^1_{k^\dagger_S}.$$

Then the k-linear homomorphism $\iota\colon k^\dagger_{S\times S} \to B\Omega^1_{k^\dagger_S}$ is independent of the choise of T and a, and coincides with the original homomorphism we have already define if S is thick. The kernel of ι contains the ideal $I_S^2 \subset k^\dagger_{S\times S}$, and hence it induces a k-linear homomorphism $\iota\colon I_S/I_S^2 \to B\Omega^1_{k^\dagger_S}$.

Definition 4.3.31. *For a k-dagger algebra A, denote by $d\colon A \to \Omega^1_A$ the universal k-linear derivation of A into finitely geberated A-modules. Set*

$$\Omega^i_A \coloneqq \textstyle\bigwedge^i_k \Omega^1_A$$

for each $i \in \mathbb{N}$.

Namely, let $I_A \subset A \otimes^\dagger_k A$ be the kernel of the multiplication $A \otimes^\dagger_k A \to A$, and set $\Omega_A \coloneqq I_A/I_A^2 = I_A \otimes_{A \otimes^\dagger_k A} A$. The $(A \otimes^\dagger_k A)$-module structure of the ideal $I_A \subset A \otimes^\dagger_k A$ and the canonical bounded k-algebra isomorphism $A \otimes^\dagger_k A/I_A \to A$ induces the A-module structure of I_A/I_A^2. The k-linear homomorphism $d\colon A \to \Omega^1_A$ is given by setting $df \coloneqq f \otimes 1 - 1 \otimes f$ mod.I_A^2. The A-module Ω^1_A is topologically generated by elements of the form df for an element $f \in A$, and furthermore Ω^1_A is algebraically finitely generated A-module. It has the following universal property: For a k-linear homomorphism $d_M\colon A \to M$ to a finitely generated A-module M such that $d_M(ab) = ad_M b + bd_M a$ for any $a, b \in A$, there is a unique A-module homomorphism $\phi\colon \Omega^1_A \to M$ such that $d_M = \phi \circ d$. See [MW] and [KLO1] for more description.

Definition 4.3.32. *Let S be a polytope, and A a k-dagger algebra. For a morphism $\gamma\colon S \to \mathscr{M}(A)$, define the pull-back homomorphism $\gamma^*\colon \Omega^1_A \to B\Omega^1_{k_S^\dagger}$ as the composition of the bounded k-linear homomorphism $I_A/I_A^2 \to I_S/I_S^2$ induced by the restriction $I_A \to I_S$ of the associated k-algebra homomorphism*

$$H^0(-, \mathbb{G}_a)(\gamma) \otimes H^0(-, \mathbb{G}_a)(\gamma)\colon A \otimes^\dagger_k A \to k_S^\dagger \otimes^\dagger_k k_S^\dagger = k_{S \times S}^\dagger$$

and the k-linear homomorphism $\iota\colon I_S/I_S^2 \to B\Omega_{k_S^\dagger}$. Note that the mage of I_A by $H^0(-, \mathbb{G}_a)(\gamma) \otimes H^0(-, \mathbb{G}_a)(\gamma)$ is obviously contained in I_S by the definitions of I_S and I_A.

Definition 4.3.33. *For a smooth k-dagger space X, denote by $\Omega^\cdot_X$ the de Rham complex of X. Call a section $\omega \in H^0(X, \Omega^i_X)$ an overconvergent differential i-form on X for an integer $i \in \mathbb{N}$.*

Definition 4.3.34. *Let S be a polytope, and X a smooth k-dagger space. For an integer $i \in \mathbb{N}$ and an element $\gamma \in Hom(S, \mathscr{A}_k^\dagger, X)$, define the pull-back homomorphism $\gamma^*\colon H^0(X, \Omega^i_X) \to H^0(S_k, B\Omega^i_S)$ as the composition of the associated k-linear homomorphism*

$$(\gamma^{(1)})^*\colon H^0(X, \Omega^i_X) \to H^0(\mathscr{M}(A_\gamma), \Omega^i_{\mathscr{M}(A_\gamma)}) = \Omega^i_{A_\gamma}$$

and the k-linear homomorphism $\Omega^i_{A_\gamma} \to B\Omega^i_{k_S^\dagger}$ induced by the k-linear homomorphism

$$(\mathscr{M}(\gamma^{(0)}))^*\colon \Omega^1_{A_\gamma} \to B\Omega^1_{k_S^\dagger}$$

by the functoriality of the wedge product, where $\mathscr{M}(\gamma^{(0)})\colon S \to \mathscr{M}(A_\gamma)$ associated with the bounded k-algebra homomorphism $\gamma^{(0)}\colon A_\gamma \to k_S$ defined in Proposition 2.4.10.

Proposition 4.3.35. *The pull-back homomorphism $\gamma^* \colon H^0(X, \Omega_X^i) \to H^0(S_k, B\Omega_S^i)$ is invariant under the equivalence relation $\sim$ on $\mathrm{Hom}(S, \mathscr{A}_k^\dagger, X)$, and hence the pull-back homomorphism $\gamma^* \colon H^0(X, \Omega_X^i) \to H^0(S_k, B\Omega_S^i)$ is well-defined for a morphism $\gamma \colon S \to X$ and an integer $i \in \mathbb{N}$.*

Proof. It is as easy calculation with commutative diagrams as we did many times in §1.4. and §2.4. □

Proposition 4.3.36. *The pull-back homomorphism $\gamma^* \colon H^0(X, \Omega_X^\cdot) \to H^0(S_k, B\Omega_S^\cdot)$ between sequences of k-linear spaces is a homomorphism of chain complices.*

Proof. Take a representative $\underline{\gamma} \in \mathrm{Hom}(S, \mathscr{A}_k^\dagger, X)$. The homomorphisms $(\mathscr{M}(\gamma^{(0)}))^*$ and $(\gamma^{(1)})^*$ are homomorphisms of chain complices, so is the composition γ^*. □

Thus we have defined the "pull-back" of an overconvergent differential form by an analytic path. The rest is the construction of the integration of an overconvergent differential form on a cube or a normalised simplex, and we deal with it in the next subsection. Concerning about a cube, the integration is very simple.

4.4 Integration of an overconvergent diffential form along a cycle

We construct the integration of an overconvergent differential form on a smooth dagger space along an analytic path in this subsection.

Definition 4.4.1. *Let $n \in \mathbb{N}$ be an integer. Define the integral*

$$\int_{[0,q_k-1]^n} \colon H^0([0, q_k - 1]_k^n, B\Omega^n_{[0,q_k-1]^n}) \to B_{dR}$$

by setting

$$\int_{[0,q_k-1]^n} \omega := \int_{[0,q_k-1]^n} f(t_1, \ldots, t_n) dt_1 \wedge \cdots \wedge dt_n \in B_{dR}$$

for a differential form $\omega = f(t_1, \ldots, t_n) dt_1 \wedge \cdots \wedge dt_n \in H^0([0, q_k - 1]_k^n, B\Omega^n_{[0,q_k-1]^n})$, where the right hand side just follows the convention of the integration of an overconvergent analytic function on $[0, q_k - 1]^n$ defined in Proposition 4.2.26.

Proposition 4.4.2. *The integral $\int_{[0,q_k-1]^n} \colon H^0([0, q_k-1]_k^n, B\Omega^n_{[0,q_k-1]^n}) \to B_{dR}$ is G_k-equivariant.*

Proof. It directly follows from Theorem 4.2.26. □

Definition 4.4.3. *For an integer $i \in \mathbb{N}$, a smooth k-dagger space X, an analytic path $\gamma \colon [0, q_k - 1]^i \to X$, and an overconvergent differential form $\omega \in H^0(X, B\Omega_X)$, define the integral of ω along γ by setting*

$$\int_\gamma \omega := \int_{[0,q_k-1]^i} \gamma^* \omega \in B_{dR}$$

We define the integral of an overconvergent differential form on the normalised simplex $(q_k - 1)\Delta^n$ in the reductive way on the dimension $n \in \mathbb{N}$. From now on, rearrange the numbering $t_1, \ldots, t_n$ of the coordinate of $\mathbb{R}^n$ to $t_0, \ldots, t_{n-1}$ and the corresponding variables. For example, the formal basis of the module $\mathrm{B}\Omega^1_{k_S^\dagger}$ of a thick polytope $S \subset \mathbb{R}^{n+1}$ is the collection $dt_0, \ldots, dt_n$.

Definition 4.4.4. *For an integer $n \in \mathbb{N}$, denote by $\mathscr{S}_{n+1}$ the symmetric group of degree $n+1$.*

To begin with, we define the integrals $\int_{(q_k-1)\Delta^0}$ and $\int_{(q_k-1)\Delta^1}$ directly. There is no difficulty here.

Definition 4.4.5. *Consider the normalised simplex $(q_k - 1)\Delta^0 = \{q_k - 1\} \subset \mathbb{R}$. The isomorphic integral affine map $0\colon \{q_k - 1\} \to 0 = \mathbb{R}^0$ induces the unique isometric isomorphism $k \to k^\dagger_{(q_k-1)\Delta^0}$. Define the integral $\int_{(q_k-1)\Delta^0}\colon H^0((q_k-1)\Delta^0_k, B\Omega^0_{(q_k-1)\Delta^0}) \to B_{dR}$ as the identity*

$$H^0((q_k-1)\Delta^0_k, B\Omega^0_{(q_k-1)\Delta^0}) = B\Omega^0_{k^\dagger_{(q_k-1)\Delta^0}} = Bk^\dagger_{(q_k-1)\Delta^0} = k \otimes_k B_{dR} \to B_{dR}.$$

This integral is trivially invariant under the actions of $T_{p,k}$ given by the canonical group homomorphism

$$T_{p,k} \hookrightarrow E_{k,1} \to (k[E_{k,1}]/(I(\Delta^0)))^\times = k^\times = (k^\dagger_{(q_k-1)\Delta^0})^\times \hookrightarrow (Bk^\dagger_{(q_k-1)\Delta^0})^\times$$

and the action of the trivial symmetric group $\mathscr{S}_1$.

Definition 4.4.6. *Consider the normalised simplex $(q_k - 1)\Delta^1 \subset \mathbb{R}^2$. The isomorphic integral affine maps $(q_k - 1)\Delta^1 \to [0, q_k - 1]\colon (t_0, t_1) \mapsto t_0$ and $(q_k - 1)\Delta^1 \to [0, q_k - 1]\colon (t_0, t_1) \mapsto t_1$ induce the two canonical isometric isomorphisms $k^\dagger_{[0,q_k-1]} \rightrightarrows k^\dagger_{(q_k-1)\Delta^1}$, and the difference between these two is the involution $*\colon k^\dagger_{(q_k-1)\Delta^1} \to k^\dagger_{(q_k-1)\Delta^1}$ defined at Definition 1.1.40 and Lemma 2.2.18. Define the integral $\int_{(q_k-1)\Delta^1}\colon H^0((q_k-1)\Delta^1_k, B\Omega^1_{(q_k-1)\Delta^1}) \to B_{dR}$ as the compositions*

$$H^0((q_k-1)\Delta^1_k, B\Omega^1_{(q_k-1)\Delta^1}) \rightrightarrows H^0([0, q_k-1]_k, B\Omega^1_{[0,q_k-1]}) \xrightarrow{\int_{[0,q_k-1]}} B_{dR}.$$

Note that these compositions detemine the same functional by Lemma 4.2.19. This integral is invariant under the action of $T^2_{p,k}$ given by the canonical group homomorphism

$$T^2_{p,k} \hookrightarrow E_{k,2} \to (k[E_{k,2}]/(I(\Delta^1)))^\times \hookrightarrow (k^\dagger_{(q_k-1)\Delta^1})^\times \hookrightarrow (Bk^\dagger_{(q_k-1)\Delta^1})^\times$$

because one has $x(q_k - 1) = (xy)(q_k - 1)$ and $\log x = \log xy$ for any $x \in E_{k,1}$ and $y \in T_{p,k}$. Moreover, this integral is invariant under the action of the symmetric group $\mathscr{S}_2$ induced by the trivial representations $\mathbb{R}^2 \times \mathscr{S}_2 \to \mathbb{R}^2$ and $\bigoplus_{i=0}^1 \mathbb{Z}dt_i \times \mathscr{S}_2 \to \bigoplus_{i=0}^1 \mathbb{Z}dt_i$ because the non-trivial element corresponds to the involution. One has $\int_{(q_k-1)\Delta^1} x dt_0 = (q_k-1)^2 x^{(0)}(q_k-1)/2 = (q_k-1)^2 x^{(1)}(q_k-1)/2$ for any $x \in E_{k,2}$ such that $x^{(0)} x^{(1)-1} \in T_{p,k}$ by the definition of $\int_0^{q_k-1}$.

Definition 4.4.7. *For integers $h \leq m \in \mathbb{N}$, denote by $dt'_{m,h} \in B\Omega^m_{k^\dagger_{(q_k-1)\Delta^m}}$ the element*

$$dt_0 \wedge \cdots \wedge dt_{h-1} \wedge dt_{h+1} \wedge \cdots \wedge dt_m.$$

For an element $x = (x^{(0)}, \ldots, x^{(m)}) \in E_{k,m+1}$, denote by $x'_{(h)} \in E_{k,m}$ the element

$$(x^{(0)}, \ldots, x^{(h-1)}, x^{(h+1)}, \ldots, x^{(m)}).$$

For integers $h_0 < h_1 \leq m \in \mathbb{N}$, denote by $dt'_{m,h_0,h_1} \in B\Omega^{m-1}_{k^\dagger_{(q_k-1)\Delta^m}}$ the element

$$dt_0 \wedge \cdots \wedge dt_{h_0-1} \wedge dt_{h_0+1} \wedge \cdots \wedge dt_{h_1-1} \wedge dt_{h_1+1} \wedge \cdots \wedge dt_m.$$

For an element $x = (x^{(0)}, \ldots, x^{(m+1)}) \in E_{k,m+2}$, denote by $x'_{(h_0,h_1)} \in E_{k,m}$ the element

$$(x^{(0)}, \ldots, x^{(h_0-1)}, x^{(h_0+1)}, \ldots, x^{(h_1-1)}, x^{(h_1+1)}, \ldots, x^{(m+1)}).$$

Now fix an integer $n \geq 2 \in \mathbb{N}_+$ and suppose the integral

$$\int_{(q_k-1)\Delta^m} : \mathrm{H}^0((q_k-1)\Delta^m_k, \mathrm{B}\Omega^m_{(q_k-1)\Delta^m}) \to \mathrm{B}_{\mathrm{dR}}$$

has already defined for each $m = 0, \ldots, n-1$. Composing the canonical B_{dR}-linear homomorphism

$$\bigoplus_{i=0}^{m} \mathrm{B}_{\mathrm{dR}}[E_{k,m+1}]dt'_{m,i} \to \bigoplus_{i=0}^{m} \mathrm{B}k^\dagger_{(q_k-1)\Delta^m} dt'_{m,i} \to \mathrm{B}\Omega^m_{k^\dagger_{(q_k-1)\Delta^m}},$$

one obtains the integral

$$\int_{(q_k-1)\Delta^m} : \bigoplus_{i=0}^{m} \mathrm{B}_{\mathrm{dR}}[E_{k,m+1}]dt'_{m,i} \to \mathrm{B}_{\mathrm{dR}}.$$

Assume this integral is invariant under the actions of $T^{m+1}_{p,k} \subset E_{k,m+1}$ given by the canonical group homomorphism

$$T^{m+1}_{p,k} \hookrightarrow E_{k,m+1} \to (k[E_{k,m+1}]/(I((q_k-1)\Delta^m)))^\times \hookrightarrow (k^\dagger_{(q_k-1)\Delta^m})^\times \hookrightarrow (\mathrm{B}k^\dagger_{(q_k-1)\Delta^m})^\times$$

and of $\mathscr{S}_{m+1}$ induced by the trivial representations $\mathbb{R}^m \times \mathscr{S}_{m+1} \to \mathbb{R}^m$ and $\bigoplus_{i=0}^m \mathbb{Z}dt_i \times \mathscr{S}_{m+1} \to \bigoplus_{i=0}^m \mathbb{Z}dt_i$. Note that the restriction $(q_k-1)\Delta^m \times \mathscr{S}_{m+1} \to (q_k-1)\Delta^m$ of the trivial representation $\mathbb{R}^m \times \mathscr{S}_{m+1} \to \mathbb{R}^m$ sends an element of $\mathscr{S}_{m+1}$ to an isomorphic integral affine map $(q_k-1)\Delta^m \to (q_k-1)\Delta^m$. In addition, assume Stokes' theorem holds for the parings $\int_{(q_k-1)\Delta^0}, \ldots, \int_{(q_k-1)\Delta^{n-1}}$, i.e. one has the equality

$$\int_{(q_k-1)\Delta^{m+1}} d(f(t_0, \ldots, t_{m+1})dt'_{m+1,h_0,h_1})$$

$$
\begin{aligned}
&= (-1)^{h_0} \int_{(q_k-1)\Delta^m} f(t_0, \ldots, t_{h_0-1}, 0, t_{h_0}, \ldots, t_m) dt'_{m,h_1-1} \\
&\quad + (-1)^{h_1} \int_{(q_k-1)\Delta^m} f(t_0, \ldots, t_{h_1-1}, 0, t_{h_1}, \ldots, t_m) dt'_{m,h_0}
\end{aligned}
$$

for integers $h_0 < h_1 \leq m < n-1 \in \mathbb{N}$ and an overconvergent analytic function $f \in \mathrm{B}k^{\dagger}_{\Delta^{m+1}}$. Furthermore, assume

$$
\int_{(q_k-1)\Delta^m} x(t_0, \ldots, t_m) dt'_{m,h} = (-1)^h \frac{(q_k-1)^m x^{(0)}(q_k-1)}{m!} = \cdots = (-1)^h \frac{(q_k-1)^m x^{(m)}(q_k-1)}{m!}
$$

for integers $h \leq m < n \in \mathbb{N}$ and an element $x \in E_{k,m+1}$ such that $x^{(i)} x^{(j)-1} \in T_{p,k}$ for any $i, j \leq m \in \mathbb{N}$.

We define the integral $\int_{(q_k-1)\Delta^n} \colon \mathrm{H}^0((q_k-1)\Delta^n_k, \mathrm{B}\Omega^n_{(q_k-1)\Delta^n}) \to \mathrm{B}_{\mathrm{dR}}$ reductively on the dimension $n \in \mathbb{N}$, and will prove the three conditions: the integral is invariant under the action of $T^{n+1}_{p,k} \times \mathscr{S}_{n+1}$, Stokes' theorem holds for the parings $\int_{(q_k-1)\Delta^0}, \ldots, \int_{(q_k-1)\Delta^n}$, and one has the equality

$$
\int_{(q_k-1)\Delta^n} x(t_0, \ldots, t_n) dt'_{n,h} = (-1)^h \frac{(q_k-1)^n x^{(0)}(q_k-1)}{n!} = \cdots = (-1)^h \frac{(q_k-1)^n x^{(n)}(q_k-1)}{n!}
$$

for an integer $h \leq n \in \mathbb{N}$ and an element $x \in E_{k,n+1}$ such that $x^{(i)} x^{(j)-1} \in T_{p,k}$ for any $i, j \leq n \in \mathbb{N}$.

Definition 4.4.8. *Take an integer $h \leq n \in \mathbb{N}$ and an element $x = (x^{(0)}, \ldots, x^{(n)}) \in E_{k,n+1}$. If $x^{(i)} x^{(j)-1} \in T_{p,k}$ for any $i < j \leq n \in \mathbb{N}$, set*

$$
\int_{(q_k-1)\Delta^n} x(t_0, \ldots, t_n) dt'_{n,h} \coloneqq (-1)^h \frac{(q_k-1)^n}{n!} \in \mathbb{Q} \subset B_{dR}.
$$

If $x^{(i)} x^{(j)-1} \notin T_{p,k}$ for some $i < j \leq n \in \mathbb{N}$, set

$$
\begin{aligned}
\int_{(q_k-1)\Delta^n} x(t_0, \ldots, t_n) dt'_{n,h}(i,j) &\coloneqq (-1)^{h+i+j} \int_{(q_k-1)\Delta^{n-1}} \frac{(-1)^i x'_{(i)} dt'_{n-1,j-1} + (-1)^j x'_{(j)} dt'_{n-1,i}}{\log x^{(i)} x^{(j)-1}} \\
&= (-1)^{h+i+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i)} - x'_{(j)}}{\log x^{(i)} x^{(j)-1}} dt'_{n-1,i} \in B_{dR}.
\end{aligned}
$$

Lemma 4.4.9. *In the latter situation above, for an element $y \in T^{n+1}_{p,k} \subset E_{k,n+1}$, one has $(xy)^{(i)} (xy)^{(j)-1} \notin T_{p,k}$ and*

$$
\int_{(q_k-1)\Delta^n} x(t_0, \ldots, t_n) dt'_{n,h}(i,j) = \int_{(q_k-1)\Delta^n} (xy)(t_0, \ldots, t_n) dt'_{n,h}(i,j)
$$

Proof. Trivial because $\log \colon \mathbb{Q}^{\vee} \to \mathrm{B}_{\mathrm{dR}}$ is invariant under the action of $T_p \subset \mathbb{Q}^{\vee}$ and $\int_{(q_k-1)\Delta^{n-1}} \colon \bigoplus_{i=0}^{n-1} \mathrm{B}_{\mathrm{dR}}[E_{k,n}] dt'_{n-1,i} \to \mathrm{B}_{\mathrm{dR}}$ is invariant under the action of $T^n_{p,k} \subset E_{k,n}$. □

Lemma 4.4.10. *The integral $\int_{(q_k-1)\Delta^n} x(t_0,\dots,t_n)dt'_{n,h}(i,j)$ is independent of the choice of the integers $i < j \le n \in \mathbb{N}$, i.e. If $x^{(i)}x^{(j)-1}, x^{(i')}x^{(j')-1} \notin T_{p,k}$ for integers $i < j \le n \in \mathbb{N}$ and $i' < j' \le n \in \mathbb{N}$, then one has the equality*

$$\int_{(q_k-1)\Delta^n} x(t_0,\dots,t_n)dt'_{n,h}(i,j) = \int_{(q_k-1)\Delta^n} x(t_0,\dots,t_n)dt'_{n,h}(i',j').$$

Hence denote by $\int_{(q_k-1)\Delta^n} x(t_0,\dots,t_n)dt'_{n,h} \in B_{dR}$ the integral.

Proof. First, consider the case $i = i'$. Set

$$\omega := \frac{(-1)^{h+j+j'}x'_{(i)}dt'_{n-1,j-1,j'-1} + (-1)^{h+i+j'}x'_{(j)}dt'_{n-1,i,j'-1} + (-1)^{h+i+j}x'_{(j')}dt'_{n-1,i,j}}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} \in \mathrm{B}\Omega^{n-2}_{k_{(q_k-1)\Delta^{n-1}}}.$$

Then one has

$$\begin{aligned}
d\omega &= \frac{(-1)^{h+j+j'}x'_{(i)}(\log x^{(j)}dt_{j-1} + \log x^{(j')}dt_{j'-1}) \wedge dt'_{n-1,j-1,j'-1}}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} \\
&\quad + \frac{(-1)^{h+i+j'}x'_{(j)}(\log x^{(i)}dt_i + \log x^{(j')}dt_{j'-1}) \wedge dt'_{n-1,i,j'-1}}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} \\
&\quad + \frac{(-1)^{h+i+j}x'_{(j')}(\log x^{(i)}dt_i + \log x^{(j)}dt_j) \wedge dt'_{n-1,i,j}}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} \\
&= \frac{(-1)^{h+i}(\log x^{(j)}x^{(j')-1})x'_{(i)}dt'_{n-1,i}}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} + \frac{(-1)^{h+i+1}(\log x^{(i)}x^{(j')-1})x'_{(j)}dt'_{n-1,i}}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} \\
&\quad + \frac{(-1)^{h+i}(\log x^{(i)}x^{(j)-1})x'_{(j')}dt'_{n-1,i}}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} \\
&= (-1)^{h+i}\left(\frac{x'_{(i)}}{\log x^{(i)}x^{(j)-1}} - \frac{x'_{(i)}}{\log x^{(i)}x^{(j')-1}} - \frac{x'_{(j)}}{\log x^{(i)}x^{(j)-1}} + \frac{x'_{(j')}}{\log x^{(i)}x^{(j')-1}}\right)dt'_{n-1,i} \\
&= (-1)^{h+i}\left(\frac{x'_{(i)} - x'_{(j)}}{\log x^{(i)}x^{(j)-1}} - \frac{x'_{(i)} - x^{(j')}}{\log x^{(i)}x^{(j')-1}}\right)dt'_{n-1,i}
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{(q_k-1)\Delta^n} x(t_0,\dots,t_n)dt'_{n,h}(i,j) - \int_{(q_k-1)\Delta^n} x(t_0,\dots,t_n)dt'_{n,h}(i,j') \\
&= (-1)^{h+i+1}\int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i)} - x'_{(j)}}{\log x^{(i)}x^{(j)-1}} - \frac{x'_{(i)} - x'_{(j')}}{\log x^{(i)}x^{(j')-1}}dt'_{n-1,i} = \int_{(q_k-1)\Delta^{n-1}} d\omega \\
&= \int_{(q_k-1)\Delta^{n-2}} \frac{(-1)^{h+j+j'}((-1)^{j-1}x'_{(i,j)}dt'_{n-2,j'-2} + (-1)^{j'-1}x'_{(i,j')}dt'_{n-2,j-1})}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}} \\
&\quad + \int_{(q_k-1)\Delta^{n-2}} \frac{(-1)^{h+i+j'}((-1)^{i}x'_{(i,j)}dt'_{n-1,j'-2} + (-1)^{j'-1}x'_{(j,j')}dt'_{n-1,i})}{\log x^{(i)}x^{(j)-1}\log x^{(i)}x^{(j')-1}}
\end{aligned}$$

$$
\begin{aligned}
& + \int_{(q_k-1)\Delta^{n-2}} \frac{(-1)^{h+i+j}((-1)^i x'_{(i,j')} dt'_{n-1,j} + (-1)^j x'_{(j,j')} dt'_{n-1,i})}{\log x^{(i)} x^{(j)-1} \log x^{(i)} x^{(j')-1}} \\
= \; & \int_{(q_k-1)\Delta^{n-2}} \frac{(-1)^{h+i+1}(x'_{(i,j)} - x'_{(i,j')}) dt'_{n-2,i}}{\log x^{(i)} x^{(j)-1} \log x^{(i)} x^{(j')-1}} - \int_{(q_k-1)\Delta^{n-2}} \frac{(-1)^{h+i+1}(x'_{(i,j)} - x'_{(j,j')}) dt'_{n-1,i}}{\log x^{(i)} x^{(j)-1} \log x^{(i)} x^{(j')-1}} \\
& + \int_{(q_k-1)\Delta^{n-2}} \frac{(-1)^{h+i+1}(x'_{(i,j')} - x'_{(j,j')}) dt'_{n-1,i}}{\log x^{(i)} x^{(j)-1} \log x^{(i)} x^{(j')-1}} \\
= \; & 0
\end{aligned}
$$

by Stokes' theorem for $\int_{(q_k-1)\Delta^{n-1}}$ and $\int_{(q_k-1)\Delta^{n-2}}$.

Secondly, consider the case $i' = j$. Since the integral $\int_{(q_k-1)\Delta^{n-1}}$ is invariant under the action of $\mathscr{S}_n$, one obtains

$$
\begin{aligned}
& \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n) dt'_{n,h}(i,j) - \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n) dt'_{n,h}(j,j') \\
= \; & (-1)^{h+i+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i)} - x'_{(j)}}{\log x^{(i)} x^{(j)-1}} dt'_{n-1,i} - (-1)^{h+j+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(j)} - x'_{(j')}}{\log x^{(j)} x^{(j')-1}} dt'_{n-1,j} \\
= \; & (-1)^{h+i+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(j)} - x'_{(i)}}{\log x^{(j)} x^{(i)-1}} dt'_{n-1,j} - (-1)^{h+i+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(j)} - x'_{(j')}}{\log x^{(j)} x^{(j')-1}} dt'_{n-1,i} \\
= \; & \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n)(x^{(i)-1} x^{(j)})(t_i)(x^{(i)} x^{(j)-1})(t_j) dt'_{n,h}(i,j) \\
& - \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n)(x^{(i)-1} x^{(j)})(t_i)(x^{(i)} x^{(j)-1})(t_j) dt'_{n,h}(i,j') \\
= \; & 0
\end{aligned}
$$

by the result in the first case.

Thirdly, consider the case $i' \neq i, j$ and $x^{(i)} x^{(i')-1} \in T_{p,k}$. Since the integral $\int_{(q_k-1)\Delta^{n-1}}$ is invariant under the action of $T^n_{p,k} \times \mathscr{S}_n$, one acquires

$$
\begin{aligned}
& \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n) dt'_{n,h}(i,j) - \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n) dt'_{n,h}(i',j') \\
= \; & (-1)^{h+i+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i)} - x'_{(j)}}{\log x^{(i)} x^{(j)-1}} dt'_{n-1,i} - (-1)^{h+i'+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i')} - x'_{(j')}}{\log x^{(i')} x^{(j')-1}} dt'_{n-1,i'} \\
= \; & (-1)^{h+i+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i)} - x'_{(j)}}{\log x^{(i)} x^{(j)-1}} dt'_{n-1,i} - (-1)^{h+i'+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i')}(x^{(i)-1} x^{(i')})(t_i) - x'_{(j')}}{\log(x^{(i)} x^{(i')-1}) x^{(i')} x^{(j')-1}} dt'_{n-1,i'} \\
= \; & (-1)^{h+i+1} \int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(i)} - x'_{(j)}}{\log x^{(i)} x^{(j)-1}} - \frac{x'_{(i)} - x'_{(j')}}{\log x^{(i)} x^{(j')-1}} dt'_{n-1,i} \\
= \; & \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n) dt'_{n,h}(i,j) - \int_{(q_k-1)\Delta^n} x(t_0, \dots, t_n) dt'_{n,h}(i,j') = 0
\end{aligned}
$$

by the result in the first case.

Finally, consider the case $i' \neq i, j$ and $x^{(i)}x^{(i')-1} \notin T_{p,k}$. One concludes

$$\int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)dt'_{n,h}(i,j) = \int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)dt'_{n,h}(i,i') = \int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)dt'_{n,h}(i',j')$$

by the result in the first and the second cases. □

Definition 4.4.11. *Define the integral*

$$\int_{(q_k-1)\Delta^n} : \bigoplus_{i=0}^{n} k[E_{k,n+1}]dt_{n,i} \to B_{dR}$$

by setting

$$\int_{(q_k-1)\Delta^n} f dt_{n,i} \coloneqq (-1)^i \sum_{x\in E_{k,n+1}} f_x \int_{(q_k-1)\Delta^n} x dt_{n,i}$$

for an integer $i \leq n \in \mathbb{N}$ and an element $f = \sum_{x\in E_{k,n+1}} f_x x \in k[E_{k,n+1}]$.

Lemma 4.4.12. *The integral $\int_{(q_k-1)\Delta^n} : \bigoplus_{i=0}^{n} k[E_{k,n+1}]dt_{n,i} \to B_{dR}$induces a k-linear homomorphism*

$$\int_{(q_k-1)\Delta^n} : \left(\bigoplus_{i=0}^{n} \left(k[E_{k,n+1}]/I((q_k-1)\Delta^n)\right) dt_{n,i}\right) / ((-1)^i dt_{n,i} - (-1)^j dt_{n,j} \mid i,j \leq n \in \mathbb{N}) \to B_{dR}.$$

Proof. It is obvious that $L((q_k-1)\Delta^n) = \mathbb{Z}(t_0+\cdots+t_n) \subset \mathbb{Z}[t_0,\ldots,t_n]$ and

$$\begin{aligned} I((q_k-1)\Delta^n) &= \sum_{x\in E_{k,1}} k[E_{k,n+1}](x(t_0+\cdots+t_n) - x(q_k-1)) \\ &= \sum_{y\in E_{k,n+1}} \sum_{x\in E_{k,1}} y(t_0,\ldots,t_n)(x(t_0+\cdots+t_n) - x(q_k-1)) \subset k[E_{k,n+1}]. \end{aligned}$$

Take an integer $h \leq n \in \mathbb{N}$, an element $y \in E_{k,n+1}$ and a character $x \in E_{k,1}$. If $y^{(i)}y^{(j)-1} \in T_{p,k}$ for any $i,j \leq n \in \mathbb{N}$, one has

$$\begin{aligned} &\int_{(q_k-1)\Delta^n} y(t_0,\ldots,t_n)(x(t_0+\cdots+t_n) - x(q_k-1))dt_{n,h} \\ =\ &\int_{(q_k-1)\Delta^n} (y^{(0)}x)(t_0)\cdots(y^{(n)}x)(t_n) - x(q_k-1)y(t_0,\ldots,t_n)dt_{n,h} \\ =\ &(-1)^h \frac{(q_k-1)^n(y^{(0)}x)(q_k-1)}{n!} - (-1)^h x(q_k-1)\frac{(q_k-1)^n y^{(0)}(q_k-1)}{n!} = 0. \end{aligned}$$

If there exists integers $i,j \leq n \in \mathbb{N}$ such that $y^{(i)}y^{(j)-1} \notin T_{p,k}$, one obtains

$$\int_{(q_k-1)\Delta^n} y(t_0,\ldots,t_n)(x(t_0+\cdots+t_n) - x(q_k-1))dt_{n,h}$$

$$\begin{aligned}
&= \int_{(q_k-1)\Delta^n} (y^{(0)}x)(t_0)\cdots(y^{(n)}x)(t_n) - x(q_k-1)y(t_0,\ldots,t_n)dt_{n,h} \\
&= (-1)^{h+i+1}\int_{(q_k-1)\Delta^{n-1}} \frac{y'_{(i)}(t_0,\ldots,t_{n-1})x(t_0+\cdots+t_{n-1}) - y'_{(j)}(t_0,\ldots,t_{n-1})x(t_0+\cdots+t_{n-1})}{\log(y^{(i)}x)(y^{(j)}x)^{-1}} dt_{n-1,i} \\
&\quad -(-1)^{h+i+1}\int_{(q_k-1)\Delta^{n-1}} x(q_k-1)\frac{y'_{(i)}-y'_{(j)}}{\log y^{(i)}y^{(j)-1}} dt_{n-1,i} \\
&= 0
\end{aligned}$$

because the image of $x(t_0+\cdots+t_{n-1}) - x(q_k-1)$ in $k^{\dagger}_{(q_k-1)\Delta^{n-1}}$ is 0. Therefore the integral induces a k-linear homomorphism

$$\int_{(q_k-1)\Delta^n} : \bigoplus_{i=0}^{n} (k[E_{k,n+1}]/I((q_k-1)\Delta^n))\, dt_{n,i} \to \mathrm{B}_{\mathrm{dR}}.$$

It sends an element of the form $(-1)^i f dt_{n,i} - (-1)^j f dt_{n,j} \in \bigoplus_{i=0}^{n} (k[E_{k,n+1}]/I((q_k-1)\Delta^n))\, dt_{n,i}$ for an element $f \in k[E_{k,n+1}]/I((q_k-1)\Delta^n)$ and integers $i, j \leq n \in \mathbb{N}$ to $0 \in \mathrm{B}_{\mathrm{dR}}$ by the definition of the integral of an element of $E_{k,n+1}$. □

Lemma 4.4.13. *For an integer $h \leq n \in \mathbb{N}$, integers $m, N \in \mathbb{N}$, and elements $x = (x_1,\ldots,x_m) \in E_{k,n}^m$, the p-adic valuations of the coefficients of the ξ-adic development of the element*

$$\xi^n \int_{(q_k-1)\Delta^n} x^I(t_0,\ldots,t_n)dt'_{n,h} \in B_{dR}^+$$

in modulo $Fil^N B_{dR}$ for a multi-index $I \in \mathbb{N}$ is of order at most $\|x\|'^I O(|I|^n)$, where $\xi \in Fil^1 B_{dR}$ is the generator of the principal ideal $Fil^1 B_{dR} \subset B_{dR}^+$ given as the logarithm of a system of p-power roots of unity.

Proof. Since

$$\xi^n \int_{(q_k-1)\Delta^n} x^I(t_0,\ldots,t_n)dt'_{n,h} = \begin{cases} (-1)^h \frac{(q_k-1)^n}{n!}\xi^n \\ \frac{(-1)^{h+i+1}\xi}{I\log x^{(i)}(x^{(j)-1}}\left(\xi^{n-1}\int_{(q_k-1)\Delta^{n-1}} (x'_{(i)})^I - (x'_{(j)})^I dt'_{n-1,i}\right) \end{cases},$$

it suffices to show that for fixed characters $y = (y_1,\ldots,y_m) \in E_{k,1}^m$, the p-adic valuations of the coefficients of the ξ-adic development of $\xi/(I\log y) \bmod .\mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}$ is of order at most $|I|$, and for fixed elements $z = (z_1,\ldots,z_m) \in E_{k,n-1}^m$, the p-adic valuations of the coefficients of the ξ-adic development of $\xi^{n-1}\int_{(q_k-1)\Delta^{n-1}} z(t_0,\ldots,t_{n-1})dt_{n-1,h} \bmod .\mathrm{Fil}^N \mathrm{B}_{\mathrm{dR}}$ is of order at most $\|z\|'^I O(|I|^{n-1})$, if I runs through all multi-indices such that $y^I \notin T_{p,k}$. Note that the integral affine maps $\Delta^{n-1} \to \Delta^n$ used in the boundary operator associates a contraction map, and one has $\|x'_{(i)}\|' \leq \|x\|'$ for an integer $i \leq n \in \mathbb{N}$ and an element $x \in E_{k,n+1}$. The first assertion has already been verified in the calculation in the proof

of the integrability of an overconvergent analytic function, Theorem 4.2.21. We show the second assertion by the induction on $m \leq n-1 \in \mathbb{N}$ of the normalised simplex. When $n = 0$, there is nothing to do. Suppose the second assertion holds for an integer $m < n-1 \in \mathbb{N}$. By Stokes' theorem, one has

$$\xi^{m+1} \int_{(q_k-1)\Delta^{m+1}} z^I(t_0, \ldots, t_{m+1}) dt'_{m+1,h} = \frac{(-1)^{h+i+1}\xi}{I \log z^{(i)}(z^{(j)-1}} \left(\xi^m \int_{(q_k-1)\Delta^m} (z'_{(i)})^I - (z'_{(j)})^I dt'_{m,i} \right)$$

for some $i, j \leq m+1 \in \mathbb{N}$ such that $(z^{(i)} z^{(j)-1})^I \notin T_{p,k}$, and hence it is of order atmost

$$O(|I|) \times \max \left\{ \|z'_{(i)}\|'^I, \|z'_{(j)}\|'^I \right\} O(|I|^m) \leq \|z\|'^I O(|I|^{m+1})$$

by the hypothesis of the induction. □

Theorem 4.4.14 (integrability of an overconvergent differential form on $(q_k-1)\Delta^n$)**.** *The integral*

$$\int_{(q_k-1)\Delta^n} : \left(\bigoplus_{i=0}^{n} (k[E_{k,n+1}]/I((q_k-1)\Delta^n))\, dt_{n,i} \right) / ((-1)^i dt_{n,i} - (-1)^j dt_{n,j} \mid i, j \leq n \in \mathbb{N}) \to B_{dR}$$

is naturally extended to a B_{dR}-linear homomorphism

$$\int_{(q_k-1)\Delta^n} : B\Omega^n_{k^\dagger_{(q_k-1)\Delta^n}} \to B_{dR}.$$

Proof. Obvious because $|F_I| \|x\|'^I$ is is of order at most $O(\delta^{|I|})$ for some parametre $\delta \in (0,1)$ for an integer $m \in \mathbb{N}$, elements $x = (x_1, \ldots, x_m) \in E^m_{k,n+1}$, and an overconvergent power series

$$F = \sum_{I \in \mathbb{N}^m} F_I T^I \in k\{\|x\|'^{-1} T\}^\dagger = k\{\|x_1\|'^{-1} T_1, \ldots, \|x_m\|'^{-1} T_m\}^\dagger.$$

□

Proposition 4.4.15 (Stoke's theorem)**.** *The integral $\int_{(q_k-1)\Delta^n} : B\Omega^n_{k^\dagger_{(q_k-1)\Delta^n}} \to B_{dR}$ satisfies the desired three conditions: the integral is invariant under the action of $T^{n+1}_{p,k} \times \mathscr{S}_{n+1}$, Stokes' theorem holds for the parings $\int_{(q_k-1)\Delta^0}, \ldots, \int_{(q_k-1)\Delta^n}$, and one has the equality*

$$\int_{(q_k-1)\Delta^n} x(t_0, \ldots, t_n) dt'_{n,h} = (-1)^h \frac{(q_k-1)^n x^{(0)}(q_k-1)}{n!} = \cdots = (-1)^h \frac{(q_k-1)^n x^{(n)}(q_k-1)}{n!}$$

for an integer $h \leq n \in \mathbb{N}$ and an element $x \in E_{k,n+1}$ such that $x^{(i)} x^{(j)-1} \in T_{p,k}$ for any $i, j \leq n \in \mathbb{N}$.

Proof. The first assertion is trivial by the definition of the integral and the hypothesis of the induction on n. The third assertion is automatically true by the definition of the integral. We verify Stokes' theorem. Take integer $h_0 < h_1 \leq n \in \mathbb{N}$ and an element $x \in E_{k,n+1}$. One has

$$\begin{aligned}
&\int_{(q_k-1)\Delta^n} d(x(t_0,\ldots,t_n)dt'_{n,h_0,h_1}) = \int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)(\log x^{(h_0)}dt_{h_0} + \log x^{(h_1)}dt_{h_1}) \wedge dt'_{n,h_0,h_1}) \\
= &\int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)((-1)^{h_0}\log x^{(h_0)}dt'_{n,h_1} + (-1)^{h_1-1}\log x^{(h_1)}dt_{n,h_0}) \\
= &(-1)^{h_1}(\log x^{(h_0)}x^{(h_1)-1})\int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)dt'_{n,h_0}.
\end{aligned}$$

If $x^{(h_0)}x^{(h_1)-1} \in T_{p,k}$, one obtains

$$(-1)^{h_1}(\log x^{(h_0)}x^{(h_1)-1})\int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)dt'_{n,h_0} = (-1)^{h_1} \times 0 \times \int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)dt'_{n,h_0} = 0$$

and

$$\begin{aligned}
&\int_{(q_k-1)\Delta^{n-1}} (-1)^{h_0}x'_{(h_0)}(t_0,\ldots,t_{n-1})dt'_{n-1,h_1-1} + (-1)^{h_1}x'_{(h_1)}(t_0,\ldots,t_{n-1})dt'_{n-1,h_0}) \\
= &(-1)^{h_1-1}\int_{(q_k-1)\Delta^{n-1}} (x'_{(h_0)} - x'_{(h_1)})dt'_{n-1,h_0} = 0
\end{aligned}$$

by the invariance if the integral $\int_{(q_k-1)\Delta^{n-1}}$ under the action of $T^n_{p,k} \times \mathscr{S}_n$. If $x^{(h_0)}x^{(h_1)-1} \notin T_{p,k}$, one obtains

$$\begin{aligned}
&(-1)^{h_1}(\log x^{(h_0)}x^{(h_1)-1})\int_{(q_k-1)\Delta^n} x(t_0,\ldots,t_n)dt'_{n,h_0} \\
= &(-1)^{h_1+1}(\log x^{(h_0)}x^{(h_1)-1})\int_{(q_k-1)\Delta^{n-1}} \frac{x'_{(h_0)} - x'_{(h_1)}}{\log x^{(h_0)}x^{(h_1)-1}}dt'_{n-1,h_0} \\
= &(-1)^{h_1+1}\int_{(q_k-1)\Delta^{n-1}} (x'_{(h_0)} - x'_{(h_1)})dt'_{n-1,h_0} \\
= &\int_{(q_k-1)\Delta^{n-1}} (-1)^{h_0}x'_{(h_0)}dt'_{n-1,h_1-1} + (-1)^{h_1}x'_{(h_1)}dt'_{n-1,h_0}.
\end{aligned}$$

□

Corollary 4.4.16. *The integral* $\int_{(q_k-1)\Delta^n} : B\Omega^n_{k^\dagger_{(q_k-1)\Delta^n}} \to B_{dR}$ *is* G_k*-equivariant.*

Proof. Using Stokes' theorem, it is reduced to the case $n = 0$, and hence trivial. □

Definition 4.4.17. *For an integer* $n \in \mathbb{N}$*, a smooth* k*-dagger space* X*, an analytic path* $\gamma\colon (q_k-1)\Delta^n \to X$*, and an overconvergent differential form* $\omega \in H^0(X, \Omega^n_X)$*, define the integral of* ω *along* γ *by setting*

$$\int_\gamma \omega := \int_{(q_k-1)\Delta^n} \gamma^*\omega \in B_{dR}.$$

4.5 Relation to Shnirel'man integral

In this subsection, we calculate basic examples of overconvergent differential forms along cycles on dagger spaces. Through the calculations, we see the relaion between our integral and Shnirel'man integral, and also find a remarkable appearence of the periods in the case of considering Tate curves. We prepared only a little explanation of the notion of the Shnirel'man integral in §0.3. If a reader is interested in Shnirel'man integral, see [SHN] for more detail.

Definition 4.5.1. *A k-dagger space X is said to be weakly contractible with respect to the analytic homology if one has*

$$H_n(X) = \begin{cases} \mathbb{Z} & (n = 0) \\ 0 & (n > 0) \end{cases} .$$

Proposition 4.5.2 (Cauchy's integral theorem)**.** *For an arbitrary integer $n \in \mathbb{N}_+$, for an overconvergent power series $f \in k\{T\}^\dagger = k\{T_1, \ldots, T_n\}^\dagger$, an analytic path $\gamma\colon [0, q_k - 1]^n \to D_k^{n\dagger}$ or $\gamma\colon (q_k - 1)\Delta^n \to D_k^{n\dagger}$ which is a cycle, one has*

$$\int_\gamma f(T_1, \ldots, T_n) dT_1 \cdots dT_n = 0.$$

In general, the integral of any closed differential form along a cycle on an weakly contractible dagger space vanishes.

Proof. The integral of a closed differential i-form $\omega \in \Omega_X^i$ along a i-cycle γ on an weakly contractible smooth dagger space X vanishes for each $i \in \mathbb{N}_+$ because there exists some $(i+1)$-cycle η on X such that $\gamma = d_{i+1}\eta$ and therefore

$$\int_\gamma \omega = \int_{d\eta} \omega = \int_\eta d\omega = \int_\eta 0 = 0.$$

The assertion is verified because $\mathrm{D}_k^{n\dagger}$ is weakly contractible by Corollary 3.2.6. □

Now incidentally, we calculate the integral $\int_\gamma f(T_1)dT_1$ according to the definition of it in practice for an integral $n = 1$, an element $f \in k\{T_1\}$, and an analytic path $\gamma\colon [0, q_k - 1] \to X$ which is a cycle. Of course it will be turned out to be 0 as we verified above, and it is just an exercise of a calculation of the integral. By Lemma morphism to a polydisc, γ corresponds to an overconvergent analytic function $g \in k_{[0,q_k-1]}^{\dagger\circ}$. Since γ is a cycle, one obtains $g(q_k - 1) = g(0) \in k$. By the construction of the correspondence, we know that $\gamma^* f = f(g) \in k_{[0,q_k-1]}^\dagger$. Therefore one has

$$\begin{aligned}
\int_\gamma f(T_1)dT_1 &= \int_{[0,q_k-1]} f(g)dg = \int_0^{q_k-1} f(g)\frac{dg}{dt_1}dt_1 = \int_0^{q_k-1} \sum_{i=0}^\infty f_i g^i \frac{dg}{dt_1} dt_1 \\
&= \int_0^{q_k-1} \sum_{i=0}^\infty \sum_{x \in E_{k,1}^i} f_i g_{x_1} \cdots g_{x_i} x_1 \cdots x_i(t_1) \sum_{x_{i+1} \in E_{k,1}} g_{x_{i+1}} (\log x_{i+1}) x_{i+1}(t_1) dt_1
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{q_k-1} \sum_{i=0}^{\infty} \sum_{x \in E_{k,1}^{i+1}} f_i g_{x_1} \cdots g_{x_{i+1}} (\log x_{i+1})(x_1 \cdots x_{i+1})(t_1) dt_1 \\
&= \int_0^{q_k-1} \sum_{i=0}^{\infty} \sum_{x \in E_{k,1}^{i+1}} f_i g_{x_1} \cdots g_{x_{i+1}} \left(\frac{1}{i+1} \sum_{j=1}^{i+1} \log x_j \right) (x_1 \cdots x_{i+1})(t_1) dt_1 \\
&= \int_0^{q_k-1} \sum_{i=0}^{\infty} \sum_{x \in E_{k,1}^{i+1}} \frac{f_i}{i+1} g_{x_1} \cdots g_{x_{i+1}} (\log x_1 \cdots x_{i+1})(x_1 \cdots x_{i+1})(t_1) dt_1 \\
&= \sum_{i=0}^{\infty} \sum_{x \in E_{k,1}^{i+1}} \frac{f_i}{i+1} g_{x_1} \cdots g_{x_{i+1}} (\log x_1 \cdots x_{i+1}) \frac{(x_1 \cdots x_{i+1})(q_k - 1) - 1}{\log x_1 \cdots x_{i+1}} \\
&= \sum_{i=0}^{\infty} \sum_{x \in E_{k,1}^{i+1}} \frac{f_i}{i+1} g_{x_1} \cdots g_{x_{i+1}} ((x_1 \cdots x_{i+1})(q_k - 1) - 1) \\
&= \sum_{i=0}^{\infty} \frac{f_i}{i+1} (g^{i+1}(q_k - 1) - g^{i+1}(0)) = \sum_{i=0}^{\infty} 0 \\
&= 0.
\end{aligned}
$$

Proposition 4.5.3 (residue theorem). *For a section $f \in H^0(\mathbb{G}_{m,k}^{\dagger}, O_{\mathbb{G}_{m,k}^{\dagger}}) \subset k((T_1))$ and an analytic path $\gamma \colon [0, q_k - 1] \to \mathbb{G}_{m,k}^{\dagger}$ which is a cycle, one has*

$$
\int_{\gamma} f(T_1) dT_1 = rot(\gamma, 0) Res(f, 0),
$$

where $rot(\gamma, 0) \in B_{dR}$ is the constant $(q_k - 1)\log xg$ given by the element $xg \in E_{k,1} \times (1 + (k_{[0,q_k-1]}^{\dagger})^{\circ\circ})$ presenting γ as axg by the unique triad $(a, x, g) \in k^{\times} \times E_{k,1} \times (1 + (k_{[0,q_k-1]}^{\dagger})^{\circ\circ})$ by the set-theoretical bijection

$$
Hom([0, q_k - 1], \mathbb{G}_{m,k}^{\dagger}) \cong k_{[0,q_k-1]}^{\dagger\times} = k^{\times} \times E_{k,1} \times (1 + (k_{[0,q_k-1]}^{\dagger})^{\circ\circ}),
$$

in Proposition 2.5.11. and $Res(f, 0)$ is the residue f_{-1} of f as an element of $k((T_1))$. In particular for an analytic path $\gamma \colon [0, q_k - 1] \to \mathbb{G}_{m,k}^{\dagger}$ given by the system $\epsilon^{1/(q_k-1)}(t_1) \in E_{k,1}$, where $\epsilon \in E_{k,1}$ is a system of power roots of unity, one has

$$
\frac{1}{\log \epsilon} \int_{\gamma} f(T_1) dT_1 = Res(f, 0).
$$

Proof. Separate f into two convergent power-series $f_{-1}T_1^{-1}$ and $f - f_{-1}T_1^{-1}$. The infinite sum

$$
F(T_1) := \sum_{i \neq -1} \frac{f_i}{i+1} T_1^{i+1} \in k((T_1))
$$

makes sense in $H^0(\mathbb{G}_{m,k}^{\dagger}, O_{\mathbb{G}_{m,k}^{\dagger}}) = \varprojlim_{r \to \infty} k\{r^{-1}T_1\}^{\dagger}$, because for any $r \in (0,\infty)$ and $f' \in k\{r^{-1}T_1\}^{\dagger}$ the infinite sum

$$\sum_{i \neq -1} \frac{f'_i}{i+1} T_1^{i+1}$$

converges in $k\{r^{-1}T_1\}^{\dagger}$. The infinite sum $F(T_1)$ is a primitive integral of $f - f_{-1}T_1^{-1}$, and hence one has

$$\int_{\gamma} (f - f_{-1}T_1^{-1}) dT_1 = \int_{\gamma} \frac{dF}{dT_1} dT_1 = \int_0^{q_k-1} \gamma^* \left(\frac{dF}{dT_1} \right) \gamma^*(dT_1) = \int_0^{q_k-1} \frac{d\gamma^*(F)}{d\gamma^*(T_1)} d\gamma^*(T_1)$$
$$= \int_0^{q_k-1} \frac{dF(axg)}{d\gamma^*(T_1)} \frac{d\gamma^*(T_1)}{dt_1} dt_1 = \int_0^{q_k-1} \frac{dF(axg)}{dt_1} dt_1 = F(ax(q_k-1)g(q_k-1)) - F(ax(0)g(0)) = 0$$

by the fundamental theorem of calculus, Proposition 4.2.32. Therefore it suffices to show the equality in the case $f = f_{-1}T_1^{-1}$. Both sides of the equality are linear on $f_{-1} = \mathrm{Res}(f,0)$, and hence we may and do assume $f = T_1^{-1}$. The one calculates

$$\begin{aligned} \int_{\gamma} T_1^{-1} dT_1 &= \int_0^{q_k-1} \gamma^*(T_1^{-1}) \gamma^*(dT_1) = \int_0^{q_k-1} (axg)^{-1} d(axg) = \int_0^{q_k-1} (axg)^{-1} axg (\log xg) dt_1 \\ &= \int_0^{q_k-1} (\log x + \log g) dt_1 = (q_k - 1) \log xg = \mathrm{rot}(\gamma, 0). \end{aligned}$$

□

Corollary 4.5.4 (Cauchy's integral formula)**.** *For an overconvergent power series $f \in k\{T_1\}^{\dagger}$, for an analytic path $\gamma \colon [0, q_k-1] \to \mathbb{D}_k^{1\dagger}$ (or $\gamma \colon [0, q_k-1] \to \mathbb{A}_k^{1\dagger}$) which is a cycle, and for a k-rational point $a \in \mathbb{D}_k^{1\dagger}(k) \backslash \gamma^{\#}([0, q_k-1]_k)$ (resp. $a \in \mathbb{A}_k^{1\dagger}(k) \backslash \gamma^{\#}([0, q_k-1]_k)$), one has*

$$\int_{\gamma} \frac{f(T_1)}{T_1 - a} dT_1 = rot(\gamma, a) f(a)$$

identifying γ as a morphism to the punctured disc $\mathbb{D}_k^{1\dagger} \backslash \{a\}$, where $rot(\gamma, a)$ is the rotation number $rot(\gamma - a, 0)$. In particular for an analytic path $\gamma \colon [0, q_k - 1] \to \mathbb{A}_k^{1\dagger}$ given by the overconvergent analytic function $\epsilon(t_1) + a \in k^{\dagger}_{[0,q_k-1]}$ for a system $\epsilon \in E_{k,1}$ of power roots of unity and an element $a \in k$, one has

$$\frac{1}{\log \epsilon} \int_{\gamma} \frac{f(T_1)}{T_1 - a} dT_1 = f(a).$$

Proof. It directly follows from the residue theorem and the presentation of the Taylor development

$$f(T_1) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{d^i f}{dT_1^i}(a) (T_1 - a)^i.$$

□

Corollary 4.5.5 (Cauchy-Goursat theorem)**.** *For an overconvergent power series $f \in k\{T_1\}^\dagger$, for an analytic path $\gamma\colon [0, q_k - 1] \to \mathbb{D}_k^{1\dagger}$ (or $\gamma\colon [0, q_k - 1] \to \mathbb{A}_k^{1\dagger}$) which is a cycle, for a k-rational point $a \in \mathbb{D}_k^{1\dagger}(k) \backslash \gamma^\sharp([0, q_k - 1]_k)$ (resp. $a \in \mathbb{A}_k^{1\dagger}(k) \backslash \gamma^\sharp([0, q_k - 1]_k)$), and for an integer $i \in \mathbb{N}_+$, one has*

$$\int_\gamma \frac{f(T_1)}{(T_1 - a)^{i+1}} dT_1 = rot(\gamma, a) \frac{d^i f}{dT_1^i}(a)$$

identifying γ as a morphism to the punctured disc $\mathbb{D}_k^{1\dagger} \backslash \{a\}$. In particular for an analytic path $\gamma\colon [0, q_k - 1] \to \mathbb{A}_k^{1\dagger}$ given by the overconvergent analytic function $\epsilon^{1/(q_k - 1)}(t_1) + a \in k_{[0, q_k - 1]}^\dagger$ for an element $a \in k$, one has

$$\frac{1}{\log \epsilon} \int_\gamma \frac{f(T_1)}{(T_1 - a)^{i+1}} dT_1 = \frac{d^i f}{dT_1^i}(a).$$

for any $i \in \mathbb{N}$.

Proof. It straightforwardly follows by the same calculation as in the proof of Cauchy's integral formula. $\square$

The element $\mathrm{rot}(\gamma, 0) \in \mathrm{B_{dR}}$ is the analogue of the rotation number in the topology theory. For example, see the follwing two characteristic cycles on $A_k^1(1, 1)^\dagger \subset \mathbb{G}_{m,k}^\dagger$, where $A_k^1(1, 1)^\dagger$ is the unit circle $\mathscr{M}(k\{T_1, T_1^{-1}\}^\dagger)$. Fix a system $\epsilon \in E_{k,1}$ of power roots of unity and consider the analytic path $\gamma_a\colon [0, q_k - 1] \to \mathbb{G}_{m,k}^\dagger$ given by the overconvergent analytic function $\epsilon^{a/(q_k - 1)} \in E_{k,1} \subset k_{[0, q_k - 1]}^\dagger$ for each $a \in \mathbb{Z}$. One has

$$\frac{1}{\log \epsilon} \int_{\gamma_a} T_1^{-1} dT_1 = \frac{1}{\log \epsilon} \mathrm{rot}(\epsilon^{a/(q_k - 1)}, 0) = a,$$

and this equality is the same one as one obtains when he or she calculates the Shnirel'man integral along a closed path of rotation number a. This is the case $g = 1$ in the definition of $\mathrm{rot}(\gamma) = (q_k - 1) \log xg$.

On the other hand, one also has a cycle with $g \neq 1$. Take an element $a \in k^{\circ\circ}$, and take systems $\underline{1 + a}, \underline{p} \in E_{k,1}$ of power roots of $1 + a$ and p respectively. There are two analytic paths connecting 1 and $1 + a$ in $\mathbb{G}_{m,k}^\dagger(k) = k^\times$: the paths γ_1 and γ_2 given by the overconvergent analyttic fuctions

$$g_1 \coloneqq \underline{1 + a}(t_1) \in E_{k,1} \subset 1 + (k_{[0, q_k - 1]}^\dagger)^{\circ\circ}$$

and

$$g_2 \coloneqq \frac{\underline{p}(t_1) - p^{q_k - 1}}{1 - p^{q_k - 1}} + \frac{1 - \underline{p}(t_1)}{1 - p^{q_k - 1}} (1 + a)^{q_k - 1} = \frac{1 - (1 + a)^{q_k - 1}}{1 - p^{q_k - 1}} \underline{p}(t_1) + \frac{(1 + a)^{q_k - 1} - p^{q_k - 1}}{1 - p^{q_k - 1}} \in 1 + (k_{[0, q_k - 1]}^\dagger)^{\circ\circ}.$$

One obtains

$$\int_{\gamma_1} T_1^{-1} dT_1 = \int_0^{q_k-1} g_1^{-1} dg_1 = \int_0^{q_k-1} (\log \underline{1+a}) dt_1 = (q_k - 1) \log \underline{1+a}$$

and

$$\begin{aligned}
\int_{\gamma_2} T_1^{-1} dT_1 &= \int_0^{q_k-1} g_2^{-1} dg_2 \\
&= \int_0^{q_k-1} \left(1 + \frac{1-(1+a)^{q_k-1}}{(1+a)^{q_k-1} - p^{q_k-1}} \underline{p}^{q_k-1}(t_1)\right)^{-1} (\log \underline{p}^{q_k-1}) \underline{p}^{q_k-1}(t_1) dt_1 \\
&= (q_k - 1)(\log \underline{p}) \int_0^{q_k-1} \sum_{i=0}^{\infty} (-1)^i \left(\frac{1-(1+a)^{q_k-1}}{(1+a)^{q_k-1} - p^{q_k-1}} \underline{p}^{q_k-1}(t_1)\right)^{i+1} dt_1 \\
&= -(q_k - 1)(\log \underline{p}) \sum_{i=0}^{\infty} \left(\frac{(1+a)^{q_k-1} - 1}{(1+a)^{q_k-1} - p^{q_k-1}}\right)^{i+1} \frac{\underline{p}^{i+1}(q_k - 1) - 1}{\log \underline{p}^{(q_k-1)(i+1)}} \\
&= -(q_k - 1)(\log \underline{p}) \sum_{i=0}^{\infty} \left(\frac{(1+a)^{q_k-1} - 1}{(1+a)^{q_k-1} - p^{q_k-1}}\right)^{i+1} \frac{p^{(q_k-1)(i+1)} - 1}{(q_k - 1)(i+1) \log \underline{p}} \\
&= -\sum_{i=0}^{\infty} \frac{1}{i+1} \left(\left(\frac{((1+a)^{q_k-1} - 1) p^{q_k-1}}{(1+a)^{q_k-1} - p^{q_k-1}}\right)^{i+1} - \left(\frac{(1+a)^{q_k-1} - 1}{(1+a)^{q_k-1} - p^{q_k-1}}\right)^{i+1}\right) \\
&= \log\left(1 - \frac{((1+a)^{q_k-1} - 1) p^{q_k-1}}{(1+a)^{q_k-1} - p^{q_k-1}}\right) - \log\left(1 - \frac{(1+a)^{q_k-1} - 1}{(1+a)^{q_k-1} - p^{q_k-1}}\right) \\
&= \log \frac{(1+a)^{q_k-1}(1 - p^{q_k-1})}{(1+a)^{q_k-1} - p^{q_k-1}} - \log \frac{1 - p^{q_k-1}}{(1+a)^{q_k-1} - p^{q_k-1}} = (q_k - 1) \log(1+a).
\end{aligned}$$

Therefore setting $\gamma \coloneqq [\gamma_1] - [\gamma_2] \in \mathrm{H}_1(\mathbb{G}_{m,k}^{\dagger})$, we conclude

$$\int_{\gamma} T_1^{-1} dT_1 = (q_k - 1) \log \underline{1+a} - (q_k - 1) \log(1+a) = (q_k - 1) \log^{\mathrm{r}} \underline{1+a}.$$

Remark that $1 + a \in 1 + (k_{[0,q_k-1]}^{\dagger})^{\circ\circ}$ and hence the reduced logarithm $\log^{\mathrm{r}} \underline{1+a}$, Definition 4.2.10, is determined without fixing a logarithm of p.

Now we calculate the period of a Tate curve $X = \mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ with a uniformiser $q \in k^{\circ\circ} \backslash \{0\}$. The two species of cycles we dealt with above are also characteristic ones of X through the embedding $A_k^1(1,1)^{\dagger} \to \mathbb{G}_{m,k}^{\dagger} \to X$. One has another cycle. Take a system $\underline{q} \in E_{k,1}$ of power roots of q. Consider the analytic path $\gamma \colon [0, q_k - 1] \to X$ determined as the composition of the analytic path $[0, q_k - 1] \to \mathbb{G}_{m,k}^{\dagger}$ given by the overconvergent analytic function $\underline{q} \in E_{k,1} \subset k_{[0,q_k-1]}^{\dagger}$ and the canonical morphism $\mathbb{G}_{m,k}^{\dagger} \twoheadrightarrow X$. The morphism γ forms a cycle on X because $\underline{q}(q_k - 1) = q^{q_k-1} \equiv 1 = \underline{q}(0) \operatorname{mod}. q^{\mathbb{Z}}$. By the way, the differential 1-form $T_1^{-1} dT_1 \in \Omega_{\mathbb{G}_{m,k}^{\dagger}}$ induce the non-trivial differential 1-form $\omega \in \Omega_X$ on X. One has

$$\int_{\gamma} \omega = \int_{\gamma'} T_1^{-1} dT_1 = \int_0^{q_k-1} \underline{q}^{-1} d\underline{q} = \int_0^{q_k-1} \log \underline{q} dt_1 = (q_k - 1) \log \underline{q},$$

and it is the period we did not find when we saw morphisms to $A^1_k(1,1)^\dagger \subset \mathbb{G}^\dagger_{m,k}$.

4.6 Pairing with coholomogies

In this subsection, we extend the integrations defined in §4.4. to pairings between homologies and cohomologies. Let X be a smooth k-dagger space.

Proposition 4.6.1 (Stokes' theorem)**.** *Consider the pairing*

$$\begin{aligned} \int : Q^\square_\cdot(X) \otimes_{\mathbb{Z}} H^0(X, \Omega^\cdot_X) &\to B_{dR} \\ \left(\sum_{j=1}^m a_j[\gamma_j]\right) \otimes \omega &\mapsto \int_{\sum_{j=1}^m a_j[\gamma_j]} \omega := \sum_{j=1}^m a_j \int_{\gamma_j} \omega \end{aligned}$$

between sequences of $\mathbb{Z}$-modules. Then the pairing satisfies Stokes' theorem, i.e. one has

$$\int_\gamma d^n \omega = \int_{d_{n+1}\gamma} \omega$$

for an integer $n \in \mathbb{N}$, a singular cube $\gamma \in Q^\square_{n+1}(X)$, and an overconvergent differential form $\omega \in H^0(X, \Omega^n_X)$.

Proof. Take an integer $n \in \mathbb{N}$, an analytic path $\gamma \colon [0, q_k - 1]^n \to X$, and an overconvergent differential form $\omega \in \mathrm{H}^0(X, \Omega^n_X)$. Since the pull-back homomorphism $\gamma^* \colon \mathrm{H}^0(X, \Omega^\cdot_X) \to \mathrm{H}^0([0,q_k]^n, \Omega^\cdot_{[0,q_k-1]^n})$ is a homomorphism of chain complices, one has

$$d^n(\gamma^*\omega) = \gamma^*(d^n\omega).$$

Present

$$\gamma^*\omega = f(t_1, \ldots, t_n) dt_1 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_n \in \mathrm{H}^0([0, q_k - 1]^n_k, \Omega^{n-1}_{[0,q_k-1]^n})$$

by a unique overconvergent analytic function $f \in k^\dagger_{[0,q_k-1]^n}$ and a unique integer $i \leq n \in \mathbb{N}_+$. Set $E_{k,n,i} \colon E_{k,i-1} \times T_{p,k} \times E_{k,n-i} \subset E_{k,n}$. Denote by $x'_{(i)}$ the element

$$x^{(1)}(t_1) \cdots x^{(i-1)}(t_{i-1}) x^{(i+1)}(t_i) \cdots x^{(n)}(t_{n-1}) \in E_{k,n-1}$$

for an element $x \in E_{k,n}$. One obtains

$$\begin{aligned} \int_\gamma d^n\omega &= \int_{[0,q_k-1]^n} \gamma^*(d_n\omega) = \int_{[0,q_k-1]^n} d^n(\gamma^*\omega) \\ &= \int_{[0,q_k-1]^n} d^n(f(t_1, \ldots, t_n) dt_1 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_n) \\ &= \int_{[0,q_k-1]^n} (-1)^{i-1} \frac{\partial f}{\partial t_i} dt_1 \wedge \cdots \wedge dt_n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in E_{k,n}} \int_{[0,q_k-1]^n} (-1)^{i-1} f_x(\log x^{(i)}) x dt_1 \wedge \cdots \wedge dt_n \\
&= \sum_{x\in E_{k,n,i}} f_x(\log x^{(i)}) \int_{[0,q_k-1]^n} x dt_1 \wedge \cdots \wedge dt_n \\
&= \sum_{x\in E_{k,n,i}} \int_{[0,q_k-1]^n} (-1)^{i-1} f_x(\log x^{(i)}) \prod_{j=1}^{n} \int_0^{q_k-1} x^{(j)}(t_1) dt_1 \\
&= \sum_{x\in E_{k,n,i}} (-1)^{i-1} f_x(\log x^{(i)}) \left(\prod_{j=1}^{i-1} \int_0^{q_k-1} x^{(j)}(t_1) dt_1 \right) \frac{x^{(i)}(q_k-1)-1}{\log x^{(i)}} \left(\prod_{j=i+1}^{n} \int_0^{q_k-1} x^{(j)}(t_1) dt_1 \right) \\
&= \sum_{x\in E_{k,n,i}} (-1)^{i-1} f_x(x^{(i)}(q_k-1)-1) \left(\prod_{j=1}^{i-1} \int_0^{q_k-1} x^{(j)}(t_1) dt_1 \right)\left(\prod_{j=i+1}^{n} \int_0^{q_k-1} x^{(j)}(t_1) dt_1 \right) \\
&= \sum_{x\in E_{k,n}} (-1)^{i-1} f_x(x^{(i)}(q_k-1)-1) \int_{[0,q_k-1]^{n-1}} x'_{(i)} dt_1 \wedge \cdots \wedge dt_{n-1} \\
&= \int_{[0,q_k-1]^{n-1}} (-1)^{i-1} f(t_1,\ldots,t_{i-1},q_k-1,t_i,\ldots,t_{n-1}) dt_1 \wedge \cdots \wedge dt_{n-1} \\
&\quad - \int_{[0,q_k-1]^{n-1}} f(t_1,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}) dt_1 \wedge \cdots \wedge dt_{n-1} \\
&= \int_{[0,q_k-1]^{n-1}} ((-1)^i(\gamma\circ\partial_n^{(i,0)})^* + (-1)^{i+1}(\gamma\circ\partial_n^{(i,1)})^*)\omega = \int_{d_{n+1}\gamma} \omega.
\end{aligned}
$$

□

Corollary 4.6.2. *Identifying $C^{\square}_{\cdot}(X) = Q^{\square}_{\cdot}(X)/D^{\square}_{\cdot}(X)$ as the $\mathbb{Z}$-submodule of $Q^{\square}_{\cdot}(X)$ generated by non-degenerate singular cubes, consider the restriction*

$$\int : C^{\square}_{\cdot}(X) \otimes_{\mathbb{Z}} H^0(X, \Omega^{\cdot}_X) \quad\to\quad B_{dR}.$$

This paring induces a G_k-equivariant canonical paring

$$\int : H^{\square}_{*}(X) \otimes_{\mathbb{Z}} H^{*}\left(H^0(X,\Omega^{\cdot}_X)\right) \to B_{dR},$$

where the action of G_k on $H^0(X,\Omega^{\cdot}_X)$ is the trivial action.

Proof. Straightforward by Stokes' theorem above. □

Proposition 4.6.3 (Stoke's theorem)**.** *Consider the pairing*

$$
\begin{aligned}
\int : C^{\triangle}_{\cdot}(X) \otimes_{\mathbb{Z}} H^0(X,\Omega^{\cdot}_X) &\quad\to\quad B_{dR} \\
\left(\sum_{j=1}^{m} a_j[\gamma_j]\right) \otimes \omega &\quad\mapsto\quad \int_{\sum_{j=1}^m a_j[\gamma_j]} \omega := \sum_{j=1}^{m} a_j \int_{\gamma_j} \omega
\end{aligned}
$$

between sequences of $\mathbb{Z}$-modules. Then the pairing satisfies Stokes' theorem, i.e. one has

$$\int_{\gamma} d^n \omega = \int_{d_{n+1}\gamma} \omega$$

for an integer $n \in \mathbb{N}$, a singular simplex $\gamma \in C^{\triangle}_{n+1}(X)$, and an overconvergent differential form $\omega \in H^0(X, \Omega^n_X)$.

Proof. Trivial by the morphism-theoretical Stokes' theorem, Proposition 4.4.15. ◻

Corollary 4.6.4. *The pairing $\int : C^{\triangle}_{\cdot}(X) \otimes_{\mathbb{Z}} H^0(X, \Omega^{\cdot}_X) \to B_{dR}$ induces a canonical G_k-equivariant paring*

$$\int : H^{\triangle}_{*}(X) \otimes_{\mathbb{Z}} H^{*}\left(H^0(X, \Omega^{\cdot}_X)\right) \to B_{dR},$$

where the action of G_k on $H^0(X, \Omega^{\cdot}_X)$ is the trivial action.

We have constructed the integral of an overconvergent differential form along a cycle in $\mathrm{H}_*(X)$ for a smooth k-dagger space X. We want to extend this integration to a pairing between analytic homologies and the de Rham cohomology. Though the de Rham cohomology of a general k-dagger space is a little complicated by the intervention of the hypercohomology, we do not have to suffer from it when we restrict the objects to the class of smooth Stein k-dagger spaces defined at Definition 2.4.27. It is well known that smooth Stein k-dagger spaces is good class to caluculate the de Rham cohomology similarly with the corresponding original Stein space in the complex geometry. See [GR] for more detail of the original one.

Lemma 4.6.5. *Let X be a Stein k-analytic space, Definition 1.4.27. or a Stein k-dagger space, Definition 2.4.27, and F an arbitrary coherent sheaf on X. Then the cohomology group $H^i(X, F)$ vanishes for any $i \in \mathbb{N}_+$.*

Proof. This is well-known fact but the proof is a little complicated. We put aside the proof to Appendix, §6.2. ◻

Corollary 4.6.6. *Let X be a smooth Stein k-algebraic variety in the sense of schemes, namely a smooth k-algebraic variety with the vanishing theorem of the higher cohomology groups of a coherent sheaf, a smooth Stein k-analytic space, or a smooth Stein k-dagger space. Then one has a canonical k-linear isomorphism*

$$H^i(H^0(X, \Omega^{\cdot}_X)) \cong_k H^i_{dR}(X)$$

for each $i \in \mathbb{N}$, where $H^*_{dR}(X)$ is the de Rham cohomology group of X. In particular, we know $H^i_{dR}(X) = 0$ for any $i > \dim_k X$.

For a precise description of the de Rham cohomology see [GRO], [BER2], and [KLO2] for an algebraic variety, an analytic space, and a dagger space respectively.

Proof. Just apply the Hodge to de Rham spectral sequence

$$E_1^{p,q} := \mathrm{H}^q(X, \Omega_X^p) \Rightarrow \mathrm{H}_{\mathrm{dR}}^{p+q}(X),$$

which is degenerated by the lemma above. Be careful about the fact that p in the spectral sequence does not indicate the characteristic of the residue field $\tilde{k}$ of k but an arbitrary integer. $\square$

Corollary 4.6.7. *Let X be a smooth Stein k-dagger space. For $H = H^\square, H^\triangle$, the pairing*

$$\int : H_*(X) \otimes_{\mathbb{Z}} H^* \left(H^0(X, \Omega_X^\cdot)\right) \to B_{dR}$$

induces a canonical pairing

$$\int : H_*(X) \otimes_{\mathbb{Z}} H_{dR}^*(X) \to B_{dR}.$$

We do not guarantee the non-degenerateness of the pairing. In fact, the homology groups $H^\square$ and $H^\triangle$ seem to have little greater dimensions than the de Rham cohomology H_{dR}^n does in general.

We have constructed a canonical pairing between analytic homologies and the de Rham cohomology for a smooth Stein dagger space. Applying it, we want to construct a G_k-equivariant pairing

$$\mathrm{H}_*(X) \otimes_{\mathbb{Z}} \mathrm{H}_{\text{ét}}^*(X, \mathbb{Q}_p) \to \mathrm{B}_{\mathrm{dR}},$$

where $\mathrm{H}_{\text{ét}}^*(X, \mathbb{Q}_p)$ is the étale cohomology. We want to consider the case X is an algebraic affine variety, i.e. the dagger space associated with an algebraic affine variety. To begin with, by the de Rham conjecture for a proper smooth algebraic variety Y derived from the semi-stable conjecture, [TSU], 4.10.4, and the semi-stable alteration, [JON], 4.5, one has a canonical G_k-equivariant isomorphism

$$\mathrm{H}_{\mathrm{dR}}^*(Y) \otimes_k \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}_{\text{ét}}^*(Y_{\overline{k}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}.$$

Be careful about that since we defined the logarithm $\log\colon \mathbb{Q}^\vee \to \mathrm{B}_{\mathrm{dR}}$ as the negative of the usual convension, the isomorphism is twisted by some sign. Furthermore, even if U is an open algebraic variety, i.e. an open subvariety of a proper smooth algebraic variety Y whose complement $D := Y \backslash U$ is a normal crossing devisor, then one has a canonical G_k-equivariant isomorphism

$$\mathrm{H}_{\mathrm{dR}}^*(U) \otimes_k \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}_{\text{ét}}^*(U_{\overline{k}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$$

by the de Rham conjecture for an open algebraic variety, [YAM]. Now suppose U is an algebraic affine variety embedded in a proper smooth variety Y whose complement

$D := Y \backslash U$ is a normal crossing devisor. Then the analytifications $U^\dagger$ and $U^{\mathrm{an}} = (U^\dagger)\hat{}$ are Stein spaces. The dagger functor $U \rightsquigarrow U^\dagger$ gives us a canonical homomorphism

$$\mathrm{H}^*_{\mathrm{dR}}(U) \to \mathrm{H}^*_{\mathrm{dR}}(U^\dagger).$$

The comparison theorem for an étale cohomology (see [BER2], 7.5.4. and [BER3], 3.2.) and the invariance of the étale cohomology of an algebraic variety under a ground field extension (see [DEL], V.3.3.) guarantee that the canonical homomorphism

$$\begin{aligned} \mathrm{H}^*_{\text{ét}}(U_{\overline{k}}, \mathbb{Z}_p) &= \varprojlim_{m\to\infty} \mathrm{H}^*_{\text{ét}}(U_{\overline{k}}, \mathbb{Z}/p^m\mathbb{Z}) \to \varprojlim_{m\to\infty} \mathrm{H}^*_{\text{ét}}(U_C, \mathbb{Z}/p^m\mathbb{Z}) \\ \to \quad & \varprojlim_{m\to\infty} \mathrm{H}^*_{\text{ét}}((U_C)^{\mathrm{an}}, \mathbb{Z}/p^m\mathbb{Z}) = \varprojlim_{m\to\infty} \mathrm{H}^*_{\text{ét}}((U^{\mathrm{an}})_C, \mathbb{Z}/p^m\mathbb{Z}) = \mathrm{H}^*_{\text{ét}}((U^{\mathrm{an}})_C, \mathbb{Z}_p) \end{aligned}$$

is a G_k-equivariant isomorphism, and it fuctors through the canonical G_k-equivariant homomorphism

$$\mathrm{H}^*_{\text{ét}}((U^\dagger)_C, \mathbb{Z}_p) = \mathrm{H}^*_{\text{ét}}((U_C)^\dagger, \mathbb{Z}_p) \to \mathrm{H}^*_{\text{ét}}(((U_C)^\dagger)\hat{}, \mathbb{Z}_p) = \mathrm{H}^*_{\text{ét}}((U_C)^{\mathrm{an}}, \mathbb{Z}_p) = \mathrm{H}^*_{\text{ét}}((U^{\mathrm{an}})_C, \mathbb{Z}_p).$$

The induced homomorphism

$$\mathrm{H}^*_{\text{ét}}(U_{\overline{k}}, \mathbb{Z}_p) \to \mathrm{H}^*_{\text{ét}}((U^\dagger)_C, \mathbb{Z}_p)$$

is a G_k-equivariant isomorphism, and tensoring B_{dR} one derives a canonical G_k-equivariant homomorphism

$$\begin{aligned} & \mathrm{H}^*_{\text{ét}}(U^\dagger, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}^*_{\text{ét}}((U^\dagger)_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \cong_{\mathrm{B}_{\mathrm{dR}}} \mathrm{H}^*_{\text{ét}}(U_{\overline{k}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \\ \cong_{\mathrm{B}_{\mathrm{dR}}} \quad & \mathrm{H}^*_{\mathrm{dR}}(U) \otimes_k \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}^*_{\mathrm{dR}}(U^\dagger) \otimes_k \mathrm{B}_{\mathrm{dR}}. \end{aligned}$$

Thus one has obtained a canonical G_k-equivariant pairing

$$\mathrm{H}_*(U^\dagger) \otimes_{\mathbb{Z}} \mathrm{H}^*_{\text{ét}}(U^\dagger, \mathbb{Q}_p) \to \mathrm{B}_{\mathrm{dR}}.$$

Recall that we defined the singular homology of U as that of $U^\dagger$. Of course this pairing might not be non-degenerated.

How about the case a dagger space X is not derived as the analytification of an affine variety? We want a canonical G_k-equivariant pairing

$$\mathrm{H}_*(X) \otimes_{\mathbb{Z}} \mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_p) \to \mathrm{B}_{\mathrm{dR}},$$

induced by some canonical G_k-equivariant pairing

$$\mathrm{C}_*(X) \otimes_{\mathbb{Z}} \mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_p) \to \mathrm{B}_{\mathrm{dR}},$$

for a dagger space X. Although we failed to construct one, we just write down what should be held in the desired pairing.

(i) Suppose a dagger space X is isomorphic to an analytification $Y^{\dagger}$ of an algebraic variety Y which is not an affine one, say, Y is a proper smooth variety. Then $Y^{\dagger}$ is proper and especially partially proper as is shown in [KLO1]. By the comparison theorems of coherent sheaves between an algebraic variety and its associated analytic space, [BER1], 3.4.10. and between a dagger space and its associated analytic space, [KLO1], 3.2, one has a canonical G_k-equivariant isomorphism

$$\mathrm{H}^*_{\mathrm{dR}}(Y) \to \mathrm{H}^*_{\mathrm{dR}}(X).$$

For an integer $n \in \mathbb{N}$, take a singular simplex $\gamma \in \mathrm{C}_n(X)$ and an element $\xi \in \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ such that the image of ξ in $\mathrm{H}^n_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ lies in the image of the canonical homomorphism

$$\begin{aligned} & \mathrm{H}^0(X, \Omega^n_X / d\Omega^{n-1}_X) \otimes_k \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}^n_{\mathrm{dR}}(X) \otimes_k \mathrm{B}_{\mathrm{dR}} \cong_{\mathrm{B}_{\mathrm{dR}}} \mathrm{H}^n_{\mathrm{dR}}(Y) \otimes_k \mathrm{B}_{\mathrm{dR}} \\ \cong_{\mathrm{B}_{\mathrm{dR}}} \quad & \mathrm{H}^n_{\text{ét}}(Y_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}^n_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}, \end{aligned}$$

and take a differential n-form $\omega \in \mathrm{H}^0(X, \Omega^n_X / d\Omega^{n-1}_X) \otimes_k \mathrm{B}_{\mathrm{dR}}$ whose image in $\mathrm{H}^n_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ is ξ. Then the value of the pairing of $\gamma \otimes \xi$ should coinced with the integral

$$\int_{\gamma} \omega \in \mathrm{B}_{\mathrm{dR}}.$$

(ii) Suppose there is a morphism $Y \to X$ from a dagger space Y with a given pairing

$$\mathrm{C}_*(Y) \otimes_{\mathbb{Z}} \mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_p) \to \mathrm{B}_{\mathrm{dR}}$$

to a dagger space X. For any $n \in \mathbb{N}$, take a singular simplex $\gamma \in \mathrm{C}_n(X)$ and an element $\xi \in \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_p)$. Suppose that the morphism $Y \to X$ satisfies the condition of a Serre fibration with respect to morphisms from a normalised cube or a normalised simplex, in other words, that any morphism η from a normalised cube or a normalised simplex to X factors the morphism $Y \to X$, or just consider the case γ is contained in the image of $\mathrm{C}_n(Y) \to \mathrm{C}_n(X)$. Then there exists a lift $\gamma' \in \mathrm{C}_n(Y)$ of γ, and the value of the pairing of $\gamma \otimes \xi$ should coincide with the value of the pairing of $\gamma' \otimes \overline{\xi}$ by the required functriality of the pairing, where $\overline{\xi} \in \mathrm{H}^n_{\text{ét}}(Y, \mathbb{Q}_p)$ is the image of ξ.

(iii) Suppose there is a morphism $X \to Y$ from a dagger space X to a dagger space Y with a given pairing

$$\mathrm{C}_*(Y) \otimes_{\mathbb{Z}} \mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_p) \to \mathrm{B}_{\mathrm{dR}}.$$

For an integer $n \in \mathbb{N}$, take a singular simplex $\gamma \in \mathrm{C}_n(X)$ and an element $\xi \in \mathrm{H}^n_{\text{ét}}(X, \mathbb{Q}_p)$. Suppose the morphism $X \to Y$ induce a surjective homomorphism $\mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_p) \to \mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_p)$, or just consider the case ξ is conitained in the image of $\mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_p) \to \mathrm{H}^*_{\text{ét}}(X, \mathbb{Q}_p)$. Then there exists a lift $\xi' \in \mathrm{H}^*_{\text{ét}}(Y, \mathbb{Q}_p)$ of ξ, and then the value of the pairing of $\gamma \otimes \xi$ coincides with the value of a pairing $\overline{\gamma} \otimes \xi'$ by the required functriality of the pairing, where $\overline{\gamma} \in \mathrm{C}_n(Y)$ is the image of γ.

For example, consider the case X is a Tate curve $\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ with a fixed uniformiser $q \in k^{\circ\circ} \backslash \{0\}$, and $Y := \mathbb{G}_{m,k}^{\dagger} \twoheadrightarrow X$ is its uniformisation. Let Z denote the proper smooth closed algebraic subvariety of the projective space $\mathbb{P}_k^2$ determined by the equality

$$T_2^2 = T_1^3 - T_1 T_2 + a_4(q) T_1 + a_6(q)$$

for suitable coefficients $a_4(q), a_6(q) \in k$, and we know that Y and X are derived from the analytifications of $\mathrm{Spec}(k[T_1, T_1^{-1}])$ and Z.

Consider the three closed paths $[0, q_k - 1] \to X$ we dealt with in the examples of the calculation at the end of §4.5: one is the path $\gamma_1 \colon [0, q_k - 1] \to X$ determined by the system $\epsilon^{1/(q_k - 1)}(t_1) \in E_{k,1} \subset (k_{[0,q_k-1]}^{\dagger})^{\circ\times}$, where $\epsilon(t_1)$ is a fixed system of power roots of unity, another one is the path $\gamma_2 \colon [0, q_k - 1] \to X$ determined by the overconvergent analytic function

$$\underline{1+a}(t_1) \left(\frac{\underline{p}(t_1) - p^{q_k - 1}}{1 - p^{q_k - 1}} + \frac{1 - \underline{p}(t_1)}{1 - p^{q_k - 1}} (1 + a)^{q_k - 1} \right)^{-1} \in (k_{[0,q_k-1]}^{\dagger})^{\circ\times}$$

for fixed systems $\underline{1+a}(t_1), \underline{p}(t_1) \in E_{k,1} \subset (k_{[0,q_k-1]}^{\dagger})^{\circ\times}$ of power roots of $1 + a$ and p respectively for an element $a \in \overline{k}^{\circ\circ}$, and the other one is the path $\gamma_3 \colon [0, q_k - 1] \to X$ determined by a fixed system $\underline{q}(t_1) \in E_{k,1} \subset (k_{[0,q_k-1]}^{\dagger})^{\circ\times}$ of power roots of the uniformiser q. They factors through the projection $Y \to X$ in a natural way. The morphisms $[0, q_k - 1] \to Y$ corresponding to γ_1 and γ_2 are again cycles, and the one corresponding to γ_3 is not a cycle. Since Z (or X) is an Abelian variety (resp. an Abelian analytic group), one has a canonical G_k-isomorphism $\mathrm{Hom}_{\mathbb{Z}_p}(T_p Z, \mathbb{Z}_p(1)) \cong_{\mathbb{Z}_p} \mathrm{H}^1_{\text{ét}}(Z_C, \mathbb{Z}_p)$ (resp. $\mathrm{Hom}_{\mathbb{Z}_p}(T_p X, \mathbb{Z}_p) \cong_{\mathbb{Z}_p} \mathrm{H}^1_{\text{ét}}(X_C, \mathbb{Z}_p)$), where $T_p Z$ (resp. $T_p X$) is the Tate's module of Z (resp. X). The canonical isomorphism

$$\begin{aligned} T_p Z \quad \cong_{\mathbb{Z}_p} \quad & \left\{ z = (z_1, z_2, \ldots) \in \varprojlim_{z_{i+1} \mapsto z_i^p} \overline{k}^{\times}/q^{\mathbb{Z}} \;\middle|\; z_1^p = 1 \in \overline{k}^{\times}/q^{\mathbb{Z}} \right\} \\ = & \left\{ z = (z'_1 q^{\mathbb{Z}}, z'_2 q^{\mathbb{Z}}, \ldots) \in \varprojlim_{z_{i+1} \mapsto z_i^p} \overline{k}^{\times}/q^{\mathbb{Z}} \;\middle|\; z_i'^{p^i} \in q^{\mathbb{Z}} \right\} \end{aligned}$$

and the well-defined evaluation map

$$\begin{array}{ccc} \varprojlim_{z_{i+1} \mapsto z_i^p} \overline{k}^{\times}/q^{\mathbb{Z}} & \to & \mathbb{Z}_p \\ z = (z'_1 q^{\mathbb{Z}}, z'_2 q^{\mathbb{Z}}, \ldots) & \mapsto & \lim_{i \to \infty} \log_{|q|} |z'_i|^{p^{i-1}} \end{array}$$

induce the splitting exact sequence

$$0 \to \mathbb{Z}_p(1) \to T_p X \to \mathbb{Z}_p \to 0$$

of $\mathbb{Z}_p$-modules. Considering the group-isomorphism $T_pX \cong_{\mathbb{Z}} \mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ given by the system $\underline{q}|_{\mathbb{Z}[p^{-1}]} \in \mathbb{Z}[p^{-1}]^\vee$ of p-power roots of q, one has a $\mathbb{Z}_p$-isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_pX, \mathbb{Z}_p) \cong_{\mathbb{Z}_p} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1), \mathbb{Z}_p) \oplus \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p, \mathbb{Z}_p) \cong_{\mathbb{Z}_p} \mathbb{Z}_p(-1) \oplus \mathbb{Z}_p,$$

and tensoring B_{dR} one obtains a B_{dR}-isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \cong_{\mathrm{B}_{\mathrm{dR}}} \mathrm{B}_{\mathrm{dR}}(-1) \oplus \mathrm{B}_{\mathrm{dR}}.$$

The isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$$

factors through the B_{dR}-isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} \to \mathrm{Hom}_{\mathbb{Z}_p}(T_pX, \mathrm{B}_{\mathrm{dR}})$$

given by the canonical non-degenerate pairing

$$\mathrm{H}^1_{\mathrm{dR}}(X) \otimes_{\mathbb{Z}_p} T_pX \to \mathbb{Z}_p.$$

Let $\xi_1, \xi_2 \in \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$ denote the B_{dR}-basis corresponding to the bases $(\epsilon|_{\mathbb{Z}[p^{-1}]} \mapsto 1) \in \mathrm{B}_{\mathrm{dR}}(-1)$ and $1 \in \mathrm{B}_{\mathrm{dR}}$. Set $\overline{\epsilon} \coloneqq \epsilon|_{\mathbb{Z}[p^{-1}]} \in \mathbb{Z}[p^{-1}]^\vee$ and $\overline{\underline{q}} \coloneqq \underline{q}|_{\mathbb{Z}[p^{-1}]} \in \mathbb{Z}[p^{-1}]^\vee$. Then the elements ξ_1, ξ_2 corresponds to the group-homomorphisms $\eta_1, \eta_2 \colon T_pX \to \mathrm{B}_{\mathrm{dR}}$ defined by

$$\begin{pmatrix} \eta_1(\overline{\epsilon}) & \eta_1(\overline{\underline{q}}) \\ \eta_2(\overline{\epsilon}) & \eta_2(\overline{\underline{q}}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{M}(2, 2; \mathrm{B}_{\mathrm{dR}}),$$

and hence one has

$$\begin{aligned}
\begin{pmatrix} (g\eta_1)(\overline{\epsilon}) & (g\eta_1)(\overline{\underline{q}}) \\ (g\eta_2)(\overline{\epsilon}) & (g\eta_2)(\overline{\underline{q}}) \end{pmatrix} &= \begin{pmatrix} g(\eta_1(g^{-1}\overline{\epsilon})) & g(\eta_1(g^{-1}\overline{\underline{q}})) \\ g(\eta_2(g^{-1}\overline{\epsilon})) & g(\eta_2(g^{-1}\overline{\underline{q}})) \end{pmatrix} \\
= \begin{pmatrix} g(\eta_1(\overline{\epsilon}^{\chi(g)^{-1}})) & g(\eta_1(\overline{\epsilon}^{\chi_{\underline{q}}(g^{-1})}\overline{\underline{q}})) \\ g(\eta_2(\overline{\epsilon}^{\chi(g)^{-1}})) & g(\eta_2(\overline{\epsilon}^{\chi_{\underline{q}}(g^{-1})}\overline{\underline{q}})) \end{pmatrix} &= \begin{pmatrix} g(\chi(g)^{-1}) & g(\chi_{\underline{q}}(g^{-1})) \\ g(0) & g(1) \end{pmatrix} \\
= \begin{pmatrix} \chi(g)^{-1} & \chi_{\underline{q}}(g^{-1}) \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} \eta_1^{\chi(g)^{-1}} \eta_2^{\chi_{\underline{q}}(g^{-1})}(\overline{\epsilon}) & \eta_1^{\chi(g)^{-1}} \eta_2^{\chi_{\underline{q}}(g^{-1})}(\overline{\underline{q}}) \\ \eta_2(\overline{\epsilon}) & \eta_2(\overline{\underline{q}}) \end{pmatrix}
\end{aligned}$$

for any $g \in G_k$, where $\chi \colon G_k \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character and $\chi_{\underline{q}} \colon G_k \to \mathbb{Z}_p$ is the set-theoretical map determined by the equality

$$g(\underline{q}(p^{-l})) = \epsilon(p^{-l})^{\chi_{\underline{q}}(g)} \underline{q}(p^{-l})$$

for each $g \in G_k$ and $l \in \mathbb{N}$. The map $\chi_{\underline{q}}$ satisfies

$$\chi_{\underline{q}}(gh) = \chi_{\underline{q}}(g) + \chi_{\underline{q}}(h)\chi(g)$$

for any $g, h \in G_k$ and in particular

$$\chi_{\underline{q}}(g^{-1}) = -\chi_{\underline{q}}(g)\chi(g)^{-1}$$

for any $g \in G_k$. Therefore it follows that

$$g\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1^{\chi(g)^{-1}} \eta_2^{\chi_{\underline{q}}(g^{-1})} \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1^{\chi(g)^{-1}} \eta_2^{-\chi_{\underline{q}}(g)\chi(g)^{-1}} \\ \eta_2 \end{pmatrix}$$

and one calculates

$$g\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \chi(g)^{-1} & -\chi_{\underline{q}}(g)\chi(g)^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Consequently one has

$$\begin{aligned} g((\log\epsilon)\xi_1 + (\log\underline{q})\xi_2) &= \chi(g)(\log\epsilon)(\chi(g)^{-1}\xi_1 - \chi_{\underline{q}}(g)\chi(g)^{-1}\xi_2) + (\log\underline{q} + \chi_{\underline{q}}(g)\log\epsilon)\xi_2 \\ &= (\log\epsilon)\xi_1 - \chi_{\underline{q}}(g)(\log\epsilon)\xi_2 + (\log\underline{q} + \chi_{\underline{q}}(g)\log\epsilon)\xi_2 \\ &= (\log\epsilon)\xi_1 + (\log\underline{q})\xi_2 \\ g(\xi_2) &= \xi_2 \end{aligned}$$

and the elements $(\log\epsilon)\xi_1 + (\log\underline{q})\xi_2$ and ξ_2 form a k-basis of the k-vector space $\mathrm{D}_{\mathrm{dR}}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}) := (\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}})^{G_k}$. The composition of the isomorphism

$$\mathrm{H}^1_{\mathrm{dR}}(X) \otimes_k \mathrm{B}_{\mathrm{dR}} \to \mathrm{H}^1_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$$

and the inclusion

$$\mathrm{H}^0(X, \Omega_X/dO_X) \otimes_k \mathrm{B}_{\mathrm{dR}} \hookrightarrow \mathrm{H}^1_{\mathrm{dR}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}$$

induced by the Hodge decomposition $\mathrm{H}^1_{\mathrm{dR}}(X) \cong_k \mathrm{H}^0(X, \Omega_X/dO_X) \oplus \mathrm{H}^1(X, O_X)$ sends the differential 1-form $T_1^{-1}dT_1 \in \mathrm{H}^0(X, \Omega_X/dO_X)$ to the element

$$(\log\epsilon)\xi_1 + (\log\underline{q})\xi_2 \in \mathrm{D}_{\mathrm{dR}}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}),$$

and therefore the pairing

$$\mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}_p) \to \mathrm{B}_{\mathrm{dR}}$$

should send $\gamma_i \otimes \xi_j$ to an element $a_{ij} \in \mathrm{B}_{\mathrm{dR}}$ for each $i = 1, \ldots, 3$ and $j = 1, 2$ such that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}\begin{pmatrix} \log\epsilon \\ \log\underline{q} \end{pmatrix} = \begin{pmatrix} \int_{\gamma_1} T_1^{-1}dT_1 \\ \int_{\gamma_2} T_1^{-1}dT_1 \\ \int_{\gamma_3} T_1^{-1}dT_1 \end{pmatrix} = \begin{pmatrix} \log\epsilon \\ (q_k - 1)\log^r(\underline{1+a}) \\ (q_k - 1)\log\underline{q} \end{pmatrix}.$$

It seems to be good if we set

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

but we have no idea of what is the desired value of $(a_{21}, a_{22}) \in (\mathrm{B}_{\mathrm{dR}})^2$ obtained from the information of the calculation above, because the path γ_2 is an irregular one for our integral theory. There is intuitively no topological hall corresponging to γ_2, but it surely has a non-zero period $\log^r(\underline{1+a})$, which is not contained in $\mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}^+$ different from other periods $\log \epsilon, \log \underline{q} \in \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}^+$.

A cycle in our homology group $\mathrm{H}_*(X)$ is namely an analytic cycle. It is defined as a homology theory concerning about analytic paths from a kind of simplicial objects. Therefore it is supposed that a singular homology concerning about algebraic paths has something to do with our singular homology. Now we want to see the relation between our singular homology and the algebraic homology of a scheme in the sense of [SV]. For a full detail of the algebraic singular homology, see [SV] and [VSF]. We follow the notation and the terminology in [VSF] chapter 6. For an arbitrary equidimensinal quasi-projective variety U of dimension $d \in \mathbb{N}$ over $\overline{k}$, denote by $z_0(U,n)$ for each $n \in \mathbb{N}$ the free Abelian group generated by an irreducible closed subvariety V of $U \otimes_{\overline{k}} \mathrm{Spec}(\overline{k}[T_0, \ldots, T_n]/(T_0 + \cdots + T_n - 1))$ such that the canonical projection $V \to \mathrm{Spec}(\overline{k}[T_0, \ldots, T_n]/(T_0 + \cdots + T_n - 1))$ is surjective and finite, and such that it properly intersects with $U \otimes_{\overline{k}} \mathrm{Spec}(\overline{k}[T_0, \ldots, T_m]/(T_0 + \cdots + T_m - 1))$ for any m-face

$$\mathrm{Spec}(\overline{k}[T_0, \ldots, T_m]/(T_0 + \cdots + T_m - 1)) \to \mathrm{Spec}(\overline{k}[T_0, \ldots, T_n]/(T_0 + \cdots + T_n - 1))$$

for any $m = 1, \ldots, n-1$ in the natural simplicial sense. Note that the free Abelian group $z_0(U,n)$ coincides with the free Abelian group generated by data of a $\overline{k}$-morphism from an irreducible affine variety over $\overline{k}$ of dimension n and suitable parametres $T_1, \ldots, T_n$ of it, by Noether's normalisation lemma. Then one can constructs a chain complex $z_0(U,*)$ by a natural way, and one has

$$\mathrm{H}_*(z_0(U,*) \otimes_{\mathbb{Z}} \mathbb{Z}/p^l\mathbb{Z}) \cong_{\mathbb{Z}/p^l\mathbb{Z}} \mathrm{CH}^d(U,*;\mathbb{Z}/p^l\mathbb{Z}) \cong_{\mathbb{Z}/p^l\mathbb{Z}} \mathrm{H}_{\text{ét}}^{2d-*}(U, \mathbb{Z}/p^l\mathbb{Z}(d))$$

for each $l \in \mathbb{N}$ by [VSF] chapter 6, 3.3. and 4.3, where $\mathrm{CH}^d(U,*;\mathbb{Z}/p^l\mathbb{Z})$ is the higher Chow group of U. See [VSF] for the higher Chow group. Suppose U is an arbitrary equidimensional affine variety of dimension $d \in \mathbb{N}$ over k embedded in a projective variety X whose complement $D \coloneqq X \backslash U$ is a normal crossing devisor, and set

$$\mathrm{H}_n^{\mathrm{sing}}(U_{\overline{k}}, \mathbb{Z}_p(-d)) \coloneqq \varprojlim_{l \to \infty} \mathrm{H}_n(z_0(U_{\overline{k}}, *) \otimes_{\mathbb{Z}} \mathbb{Z}/p^l$$

for each $n \in \mathbb{N}$. By the argument above, one obtains a canonical pairing

$$\mathrm{H}_*(U) \otimes_{\mathbb{Z}} \mathrm{H}_{2d-*}^{\mathrm{sing}}(U_{\overline{k}}, \mathbb{Z}_p(-d)) \to \mathrm{B}_{\mathrm{dR}}.$$

This is a kind of a cup product of an analytic cycle and an algebraic cycle whose sum of the dimensions is $2d$.

5 Examples

Though many of the following calculations are valid even if k is not a local field, we continue the assumption that k is a local field for convenience. In this section, we calculate the singular homology group of basic k-dagger spaces. The main process of the calculation consists of three steps. The first step is determining the set of the morphisms from a normalised cube $[0, q_k - 1]^n$ and a normalised simplex $(q_k - 1)\Delta^n$. It is hard in general, but the adjoint property, Proposition 2.4.10, and its corollaries of it will help us in the calculation. The second step is determining the subgroup of cycles in $\mathrm{C}_n(\cdot)$. It is also hard in general, but Corollary 3.2.24. will be useful when a dagger space is contained in an analytic group as an analytic domain. The final step is determining the subgroup of boundaries. It is the most difficult step, but Stokes' theorem, Proposition 4.6.1. and Proposition 4.6.3, for the integral of a differential form guarantees a necessary condition for a cycle not to be a boundary, i.e. the integral of any differential form along a boundary vanishes. To begin with, we recall the basic result of the application of the homotopy invariance of the singular homology, Proposition 3.2.5.

Corollary 3.2.6. *For integers $n, m \in \mathbb{N}$ and $d \in (0, \infty)^m$, one has*

$$\begin{aligned} H_n(\mathring{D}^m_k(d)) \cong_{\mathbb{Z}} H_n(D^m_k(d)) \cong_{\mathbb{Z}} H_n(\mathbb{A}^m_k) &\cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & (n = 0) \\ 0 & (n > 0) \end{cases} \\ H_n(\mathring{D}^m_k(d)^\dagger) \cong_{\mathbb{Z}} H_n(D^m_k(d)^\dagger) \cong_{\mathbb{Z}} H_n(\mathbb{A}^{m\dagger}_k) &\cong_{\mathbb{Z}} \begin{cases} \mathbb{Z} & (n = 0) \\ 0 & (n > 0) \end{cases} . \end{aligned}$$

We see four other basic examples: the homology group of the punctured affine line $\mathbb{G}^\dagger_{m,k} = \mathbb{A}^{1\dagger}_k \backslash \{0\}$, a punctured disc $\mathrm{D}^{1\dagger}_k \backslash \{a_1, \ldots, a_m\}$, a Tate curve $\mathbb{G}^\dagger_{m,k}/q^{\mathbb{Z}}$, and the projective line $\mathbb{P}^{1\dagger}_k$. We see the first example right now.

Proposition 5.1.1. *For $X = A^1_k(1,1)^\dagger$ or $\mathbb{G}^\dagger_{m,k}$, one has a canonical $\mathbb{Z}[G_k]$-module isomorphism*

$$H_0(X) \cong_{\mathbb{Z}[G_k]} \mathbb{Z},$$

where the action of G_k on $\mathbb{Z}$ in the right hand side is trivial, and a canonical exact sequence

$$0 \to \hat{\mathbb{Z}}(1) \to H_1(X) \to 1 + k^{\circ\circ} \to 0$$

of $\mathbb{Z}[G_k]$-modules.

Note that this proposition is verified with no use of the integration. Therefore the value of N_k is not restricted to be $q_k - 1$ or X is not restricted to be a dagger space. We prove the assertion only in the case $N_k = q_k - 1$ and X is a dagger space, and there is no difference in the proof also in the case $N_k = 1$ or X is an analytic space.

Proof. By Proposition 3.2.18, it suffices to show the analytically pathwise connectedness of X in order to verify the first assertion. Take a k-rational point $a \in X(k)$. We verify that a shares the analyticall pathwise connected component with $1 \in X(k)$. Take a system $\underline{a} \in E_{k,1}$ of power roots of a, and then one has

$$\begin{aligned} \|\underline{a}\| &= \max\{1, |a|\} \\ \text{and } \|\underline{a}^{-1}\| &= \max\{1, |a|^{-1}\}. \end{aligned}$$

If $X = \mathrm{A}^1_k(1,1)^\dagger$, then $a \in k^{\circ\times}$ and hence $\|\underline{a}\| = \|\underline{a}^{-1}\| = 1$. Therefore regardless of whether $X = \mathrm{A}^1_k(1,1)^\dagger$ or $\mathbb{G}^\dagger_{m,k}$, the character $\underline{a} \in E_{k,1} \subset (k^\dagger_{[0,q_k-1]})^\times$ determines a morphism $[0, q_k - 1] \to X$, which connects a and 1. Consequently X is analytically pathwise connected and one has

$$\mathrm{H}_0(X) \cong_{\mathbb{Z}} \mathbb{Z}.$$

Since X is an analytically pathwise connected analytic group, the set-theoretical map $\mathrm{Hom}([0, q_k - 1], X) \to \mathrm{C}_1(X) \to \mathrm{H}_1(X)$ is surjective by Corollary 3.2.24. Set $L := \{(x, 1 + g) \in E_1 \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ}) \mid x(1)(1 + g(1)) = 1\}$, where E_1 is the subgroup $\{x \in E_{k,1} \mid |x(1)| = 1\}$. We show that the restriction $L \to \mathrm{H}_1(X)$ of the set-theoretical map

$$L \subset 1 \times E_1 \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ}) \subset [\tilde{C}^\times] \times E_1 \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ}) \to \mathrm{Hom}([0, q_k - 1], X) \to \mathrm{H}_1(X)$$

is a surjective group-homomorphism.

Consider the case $X = \mathrm{A}^1_k(1,1)^\dagger$ first. One has the set-theoretical map

$$[\tilde{C}^\times] \times E_1 \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ}) \to \mathrm{Hom}([0, q_k - 1], X)$$

is bijective. Take a cycle $\gamma \in \mathrm{H}_1(X)$. Since $\mathrm{Hom}([0, q_k - 1], X) \to \mathrm{H}_1(X)$ is surjective, there exists an analytic path $\eta \in \mathrm{Hom}([0, q_k - 1], X)$ such that $[\eta] = \gamma \in \mathrm{H}_1(X)$. Let $(a, x, 1 + g) \in [\tilde{C}^\times] \times E_1 \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ})$ be the pair corresponding to η. Since η is a cycle, one has $x(q_k - 1)(1 + g(q_k - 1)) = x(0)(1 + g(0)) = 1 + g(0)$. Consider the analytic path $\xi \in \mathrm{Hom}([0, q_k - 1], X)$ associated with the bounded k-homomorphism $k\{T_1, T_1^{-1}\} \to k^\dagger_{[0,q_k-1]} \colon T_1 \to a^{-1} g(0)^{-1}$. This is a constant morphism and hence a cycle. One knows $[\xi] = 0 \in \mathrm{H}_1(X)$ by Lemma 3.2.20. Since the group structure of $\mathrm{H}_1(X)$ is compatible with that induced by the structure of X as a group object, the cycle determined by the pair $(1, x, (1+g)g(0)^{-1}) \in [\tilde{C}^\times] \times E_1 \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ})$ coincides with γ as an element of $\mathrm{H}_1(X)$. Therefore replacing a to 1 and $1 + g$ to $(1 + g)g(0)^{-1} = 1 + (g(0)^{-1} - 1) + g \in k^{\dagger\circ\circ}_{[0,q_k-1]}$, we may assume $a = 1$ and $(x, 1 + g) \in L$. It follows that the set-theoretical map $L \to \mathrm{H}_1(X)$ is surjective, and it is a group-homomorphism because the group structure of $\mathrm{H}_1(X)$ is compatible with that induced by the structure of X as a group object.

On the other hand, suppose $X = \mathbb{G}^\dagger_{m,k}$. The set-theoretical map

$$[\tilde{C}^\times] \times E_1 \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ}) \to \mathrm{Hom}([0, q_k - 1], X)$$

factors through the isomorphism

$$[\tilde{C}^{\times}] \times \pi_k^{\mathbb{Z}} \times E_1 \times (1 + (k^{\dagger}_{[0,q_k-1]})^{\circ\circ}) \to \mathrm{Hom}([0, q_k - 1], X),$$

where $\pi_k \in k$ is a uniformiser. However, since an element in $\pi_k^{\mathbb{Z}}$ corresponds to a constant morphism, any morphism determined by an element of $[\tilde{C}^{\times}] \times \pi_k^{\mathbb{Z}} \times E_1 \times (1 + (k^{\dagger}_{[0,q_k-1]})^{\circ\circ})$ can be derived from $[\tilde{C}^{\times}] \times E_1 \times (1 + (k^{\dagger}_{[0,q_k-1]})^{\circ\circ})$ by the same argument as above. Therefore also in this case, the set-theoretical map $L \to \mathrm{H}_1(X)$ is a surjective group-homomorphism.

Now set $L' := \{x \in E_{k,1} \mid x(q_k - 1) \in 1 + \overline{k}^{\circ\circ}\}$. In both cases, we prove that the group-homomorphism $L \to \mathrm{H}_1(X)$ factors the canonical projection $L \to L'$. Take an arbitrary element $(x, 1 + g) \in L$ in the kernel of $L \to L'$, and then $x = 1$. By the definition of L, one has $g(0) = g(q_k - 1) = 0$. Since $\|(1 + g) - 1\| = \|g\| < 1$, the morphism $\gamma \in \mathrm{Hom}([0, q_k - 1], X)$ uniquely factors the analytic domain $1 + \mathring{\mathrm{D}}^1_k(1)^{\dagger} \subset X$, and there exists some $\zeta \in \mathrm{C}_2(\mathring{\mathrm{D}}^1_k(1)^{\dagger})$ such that $[\gamma] = d\zeta \in \mathrm{C}_1(\mathring{\mathrm{D}}^1_k(1)^{\dagger})$ because $\mathrm{H}_1(\mathring{\mathrm{D}}^1_k(1)^{\dagger}) = 0$. Then one also has $[\gamma] = d\zeta \in \mathrm{C}_1(X)$, and hence $[\gamma] = 0 \in \mathrm{H}_1(X)$. It implies that $L \to \mathrm{H}_1(X)$ uniquely factors $L \to L'$. The well-defined group-homomorphism $L' \to \mathrm{H}_1(X)$ is presented as

$$\begin{array}{rcl} L' & \to & \mathrm{H}_1(X) \\ x & \mapsto & \gamma_x := \left[x(t_1) \left(-\dfrac{x(q_k - 1) - 1}{1 - p} \underline{p}(t_1) + \dfrac{x(q_k - 1) - p}{1 - p} \right)^{-1} \right]. \end{array}$$

Next, we verify that the group homomorphism $L' \to \mathrm{H}_1(X)$ is injective and hence bijective. Since $L' \to \mathrm{H}_1(\mathbb{G}^{\dagger}_{m,k})$ factors through the homomorphism $\mathrm{H}_1(\mathrm{A}^1_k(1,1)^{\dagger}) \to \mathrm{H}_1(\mathbb{G}_{m,k})$ associated with the inclusion $\mathrm{A}^1_k(1,1)^{\dagger} \hookrightarrow \mathbb{G}^{\dagger}_{m,k}$ by the definitions of the homomorphisms $L'\mathrm{H}_1(\mathrm{A}^1_k(1,1)^{\dagger})$ and $L' \to \mathrm{H}_1(\mathbb{G}_{m,k})$, it is enough to consider the case $X = \mathbb{G}_{m,k}$. We calculate the image $d_2\mathrm{C}_2(X) \subset \mathrm{C}_1(X)$. Consider the case $\mathrm{C} = \mathrm{C}^{\square}$. There are the canonical set-theoretical bijective maps

$$k^{\times} \times E_{k,1} \times (1 + (k^{\dagger}_{[0,q_k-1]})^{\circ\circ}) \to (k^{\dagger}_{[0,q_k-1]})^{\times} \to \mathrm{Hom}([0, q_k - 1], X)$$

and

$$k^{\times} \times E_{k,2} \times (1 + (k^{\dagger}_{[0,q_k-1]^2})^{\circ\circ}) \to (k^{\dagger}_{[0,q_k-1]^2})^{\times} \to \mathrm{Hom}([0, q_k - 1]^2, X).$$

The group $\mathrm{C}^{\square}_1(X)$ is the free $\mathbb{Z}$-module generated by non-constant analytic paths $[0, q_k - 1] \to X$. The identification $k^{\times} \times E_{k,1} \times (1 + (k^{\dagger}_{[0,q_k-1]})^{\circ\circ}) = \mathrm{Hom}([0, q_k - 1], X)$ induces a group homomorphism $\mathrm{C}^{\square}_1(X) \to E_{k,1}$. The restriction $\phi\colon d_2\mathrm{C}^{\square}_2(X) \to E_{k,1}$ is the zero homomorphism 1. Indeed, take a non-degenerate analytic path $\gamma\colon [0, q_k - 1]^2 \to X$ associated a pair $(b, y, (1 + h)) \in k^{\times} \times E_{k,2} \times (1 + (k^{\dagger}_{[0,q_k-1]^2})^{\circ\circ})$. One has

$$d_2[\gamma] = d_2\left[by^{(1)}(t_1)y^{(2)}(t_2)(1 + g(t_1, t_2)) \right]$$

$$\begin{aligned} &= -\left[by^{(2)}(t_1)(1+g(0,t_1))\right] + \left[by^{(1)}(q_k-1)y^{(2)}(t_1)(1+g(q_k-1,t_1))\right] \\ &\quad + \left[by^{(1)}(t_1)(1+g(t_1,0))\right] - \left[by^{(1)}(q_k-1)y^{(1)}(t_1)(1+g(t_1,q_k-1))\right] \end{aligned}$$

and hence

$$\phi(d_2[\gamma]) = (y^{(2)})^{-1} y^{(2)}(t_1) y^{(1)} (y^{(1)}(t_1))^{-1} = 1.$$

Therefore for any character $x \in L'$ in the kernel of $L' \to \mathrm{H}_1^{\square}(X)$, taking a lift $(x, 1+g) \in L$, one obtains $[x(1+g)] \in d_2 \mathrm{C}_2^{\square}(X)$ and hence $x = 1$. It follows $L' \to \mathrm{H}_1^{\square}(X)$ is injective. Consider the case $\mathrm{C} = \mathrm{C}^{\triangle}$. Taking the isomorphic integral affine maps

$$(q_k-1)\Delta^1 \to [0, q_k-1] \colon (t_0, t_1) \to t_0$$

and

$$(q_k-1)\Delta^2 \to \triangle := \left\{ (t_1,t_2) \in \mathbb{R}^2 \,\middle|\, t_1 \geq 0, t_1 \geq 0, t_1 + t_2 \leq q_k - 1 \right\} \colon (t_0,t_1,t_2) \to (t_0,t_1),$$

one obtains the canonical set-theoretical bijective maps

$$k^{\times} \times E_{k,1} \times (1 + (k^{\dagger}_{[0,q_k-1]})^{\circ\circ}) \to (k^{\dagger}_{[0,q_k-1]})^{\times} \to (k^{\dagger}_{(q_k-1)\Delta^1})^{\times} \to \mathrm{Hom}((0,q_k-1)\Delta^1, X)$$

and

$$k^{\times} \times E_{k,2} \times (1 + (k^{\dagger}_{\triangle})^{\circ\circ}) \to (k^{\dagger}_{\triangle})^{\times} \to (k^{\dagger}_{(q_k-1)\Delta^2})^{\times} \to \mathrm{Hom}((q_k-1)\Delta^2, X).$$

The group $\mathrm{C}_1^{\triangle}(X)$ is the free $\mathbb{Z}$-module generated by non-constant analytic paths $(q_k - 1)\Delta^1 \to X$. The identification $k^{\times} \times E_{k,1} \times (1 + (k^{\dagger}_{[0,q_k-1]})^{\circ\circ}) = \mathrm{Hom}((q_k-1)\Delta^1, X)$ induces a group homomorphism $\mathrm{C}_1^{\triangle}(X) \to E_{k,1}$. The restriction $\phi \colon d_2 \mathrm{C}_2^{\triangle}(X) \to E_{k,1}$ is the zero homomorphism 1. Indeed, take an analytic path $\gamma \colon \Delta^2 \to X$ associated a pair $(b, y, (1+h)) \in k^{\times} \times E_{k,2} \times (1 + (k^{\dagger}_{\triangle})^{\circ\circ})$. One has

$$\begin{aligned} d_2[\gamma] &= d_2\left[by^{(1)}(t_1)y^{(2)}(t_2)(1+g(t_1,t_2))\right] \\ &= \left[by^{(2)}(t_1)(1+g(0,t_1))\right] - \left[by^{(1)}(t_1)(1+g(t_1,0))\right] \\ &\quad + \left[by^{(2)}(q_k-1)\left(y^{(1)}y^{(2)-1}\right)(t_1)(1+g(t_1,1-t_1))\right] \end{aligned}$$

and hence

$$\phi(d_2[\gamma]) = y^{(2)}(y^{(1)})^{-1}(y^{(1)}y^{(2)-1}) = 1.$$

Therefore for any character $x \in L'$ in the kernel of $L' \to \mathrm{H}_1^{\triangle}(X)$, taking a lift $(x, 1+g) \in L$, one obtains $[x(1+g)] \in d_2 \mathrm{C}_2^{\triangle}(X)$ and hence $x = 1$. It follows $L' \to \mathrm{H}_1^{\triangle}(X)$ is injective. We have verified that $L' \cong \mathrm{H}_1(X)$.

Finally, cosider the evaluation homomorphism $L' \to 1 + k^{\circ\circ} \colon x \mapsto x(q_k - 1)$. It is surjective, and its kernel is $(q_k-1)^{-1}\hat{\mathbb{Z}}(1) \subset E_{k,1} \cap \hat{\mathbb{Z}}(1) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathbb{Q}^{\vee}$, and the canonical isomorphism $(q_k-1)^{-1}\hat{\mathbb{Z}}(1) \to \hat{\mathbb{Z}}(1)$ gives the canonical exact sequence

$$0 \to \hat{\mathbb{Z}}(1) \to L' \to 1 + k^{\circ\circ} \to 0$$

and the G_k-equivariance holds because so is each deformation of formulae above. $\square$

Corollary 5.1.2. *In the situation above, one has a canonical isomorphism*

$$H_0(X, \mathbb{Q}_p) \cong_{\mathbb{Q}_p[G_k]} \mathbb{Q}_p$$

and a canonical exact sequence

$$0 \to \mathbb{Q}_p(1) \to H_1(X, \mathbb{Q}_p) \to k \to 0$$

of p-adic Galois representations.

Be careful that $\mathrm{H}_*(\cdot, \mathbb{Q}_p)$ is the analytic homology with coefficients in $\mathbb{Q}_p$ as a pro-object, which we defined at Definition 3.1.7, but is not the analytic homology with coefficients in $\mathbb{Q}_p$ as a group.

Proof. The first isomorphism $\mathrm{H}_0(X, \mathbb{Q}_p) \cong_{\mathbb{Q}_p[G_k]} \mathbb{Q}_p$ is trivial. By universal coefficient theorem, Proposition 3.2.2, one has the canonical exact sequence

$$0 \to \mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to \mathrm{H}_1(X, \mathbb{Z}/p^m\mathbb{Z}) \to \mathrm{Tor}^{\mathbb{Z}}(\mathrm{H}_0(X), \mathbb{Z}/p^m\mathbb{Z}) \to 0$$

of $\mathbb{Z}[G_k]$-modules for each $m \in \mathbb{N}$. Since $\mathrm{H}_0(X) \cong_{\mathbb{Z}} \mathbb{Z}$ is a torsionfree $\mathbb{Z}$-module, one obtains $\mathrm{Tor}^{\mathbb{Z}}(\mathrm{H}_0(X), \mathbb{Z}/p^m\mathbb{Z}) = 0$ and hence $\mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \cong_{\mathbb{Z}[G_k]} \mathrm{H}_1(X, \mathbb{Z}/p^m\mathbb{Z})$. It follows

$$\mathrm{H}_1(X, \mathbb{Q}_p) \;=\; \left(\varprojlim_{m\to\infty} \mathrm{H}_1(X, \mathbb{Z}/p^m\mathbb{Z})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong_{\mathbb{Q}_p[G_k]} \left(\varprojlim_{m\to\infty} \mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z}\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Tensoring $\mathbb{Z}/p^m\mathbb{Z}$ to the short exact sequence

$$0 \to \hat{\mathbb{Z}}(1) \to \mathrm{H}_1(X) \to 1 + k^{\circ\circ} \to 0,$$

one acquires the long exact sequence

$$\cdots \to \mathrm{Tor}^{\mathbb{Z}}(1 + k^{\circ\circ}, \mathbb{Z}/p^m\mathbb{Z}) \to \hat{\mathbb{Z}}(1) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to \mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to (1 + k^{\circ\circ}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to 0.$$

Let $M \in \mathbb{N}$ be the greatest integer such that k has a primitive p^M-th root of unity, and suppose $m \geq M \in \mathbb{N}$. Then one obtains $\mathrm{Tor}^{\mathbb{Z}}(1+k^{\circ\circ}, \mathbb{Z}/p^m\mathbb{Z}) = \mathbb{Z}/p^M\mathbb{Z}(1) \subset 1+k^{\circ\circ}$. Moreover one has $\hat{\mathbb{Z}}(1) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} = \mathbb{Z}/p^m\mathbb{Z}(1)$, and the homomorphism $\mathrm{Tor}^{\mathbb{Z}}(1 + k^{\circ\circ}, \mathbb{Z}/p^m\mathbb{Z}) \to \hat{\mathbb{Z}}(1) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z}$ coincides with the canonical embedding $\mathbb{Z}/p^M\mathbb{Z}(1) \hookrightarrow \mathbb{Z}/p^m\mathbb{Z}(1)$ given by tensoring $\hat{\mathbb{Z}}(1)$ to the inclusion $p^{-M}\mathbb{Z}/\mathbb{Z} \hookrightarrow p^{-m}\mathbb{Z}/\mathbb{Z}(1) \subset \mathbb{Q}/\mathbb{Z}$ through the identifications $\mathbb{Z}/p^M\mathbb{Z} \cong_{\mathbb{Z}} p^{-M}\mathbb{Z}/\mathbb{Z}$ and $\mathbb{Z}/p^m\mathbb{Z} \cong_{\mathbb{Z}} p^{-m}\mathbb{Z}/\mathbb{Z}$. Therefore its cokernel is $\mathbb{Z}/p^{m-M}\mathbb{Z}(1)$, and one acquires the exact sequence

$$0 \to \mathbb{Z}/p^{m-M}\mathbb{Z}(1) \to \mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to (1 + k^{\circ\circ}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to 0.$$

It induces the exact sequence

$$\begin{array}{ccccccccc}
 & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/p^{m+1-M}\mathbb{Z}(1) & \longrightarrow & \mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^{m+1}\mathbb{Z} & \longrightarrow & (1+k^{\circ\circ}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^{m+1}\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/p^{m-M}\mathbb{Z}(1) & \longrightarrow & \mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^{m}\mathbb{Z} & \longrightarrow & (1+k^{\circ\circ}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^{m}\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow & &
\end{array}$$

of chain complices, which obviously satisfies Mittag-Leffler condition, and hence one obtains the exact sequence

$$0 \to \mathbb{Z}_p(1) \to \varprojlim_{m\to\infty} \mathrm{H}_1(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to \varprojlim_{m\to\infty} (1+k^{\circ\circ}) \otimes_{\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \to 0.$$

Since the higher unit group $1+k^{\circ\circ}$ is p-adically complete, the third term coincides with $1+k^{\circ\circ}$. Furthermore, since $\mathbb{Q}_p$ is a flat $\mathbb{Z}_p$-algebra, it gives the exact sequence

$$0 \to \mathbb{Q}_p(1) \to \mathrm{H}_1(X, \mathbb{Q}_p) \to (1+k^{\circ\circ}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$

Now consider the logarithm homomorphism $\log\colon 1+k^{\circ\circ} \to k$. Since its kernel is the torsion subgroup $\mathbb{Z}/p^M\mathbb{Z}(1) \subset 1+k^{\circ\circ}$ and since its image contains the convergent domain of the exponential map, it induces the canonical isomorphism $(1+k^{\circ\circ}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to k$. Thus one acquires the canonical exact sequence

$$0 \to \mathbb{Q}_p(1) \to \mathrm{H}_1(X, \mathbb{Q}_p) \to k \to 0,$$

and the G_k-equivariance holds because so is each deformation of formulae above. □

Corollary 5.1.3. *In the situation above, one has canonical isomorphisms*

$$H_i(X, \mathbb{Q}_l) \cong_{\mathbb{Q}_l} \begin{cases} \mathbb{Q}_l & (i=0) \\ \mathbb{Q}_l(1) & (i=1) \end{cases}$$

of l-adic representations for a prime number $l \neq p \in \mathbb{N}$.

Proof. The isomorphisms are constructed in the totally same way as in the case $l = p$. Just remark that $1+k^{\circ\circ}$ is l-torsionfree and l-divisible. □

Corollary 5.1.4. *In the situation above, the p-adic analytic homology group $H_i(X, \mathbb{Q}_p)$ is a crystalline representation for $i = 0, 1$.*

Proof. Trivial for $\mathrm{H}_0(X, \mathbb{Q}_p)$. Identify $\mathrm{H}_1(X, \mathbb{Q}_p)$ as the $\mathbb{Q}_p[G_k]$-module of the tensor $\mathbb{Q}_p$ of the $\mathbb{Z}_p$-module of systems of p-power roots of elements of $1+k^{\circ\circ}$ by the proof of Corollary 5.1.2. Then the Galois invariants $(\mathrm{H}_1(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{crys}})^{G_k}$ admits a k-basis

$$\epsilon \otimes \frac{1}{\log \epsilon}, \left(\underline{\exp e_1} \otimes 1\right) - \left(\epsilon \otimes \frac{\log \underline{\underline{\exp e_1}}}{\log \epsilon}\right), \ldots, \left(\underline{\exp e_{\dim_{\mathbb{Q}_p} k}} \otimes 1\right) - \left(\epsilon \otimes \frac{\log \underline{\underline{\exp e_{\dim_{\mathbb{Q}_p} k}}}}{\log \epsilon}\right)$$

where $\epsilon \in E_{k,1}$ is a system of p-power roots of unity, $\{e_1, \ldots, e_{\dim_{\mathbb{Q}_p} k}\} \subset k$ is a $\mathbb{Q}_p$-basis consisting of elements in the convergent domain of the exponential map, $\underline{\exp e_i} \in E_{k,1}$ is a system of p-power roots of $\exp e_i \in 1 + k^{\circ\circ}$ for each $i = 1, \ldots, \dim_{\mathbb{Q}_p} k$. ◻

The results are remarkable. Each l-adic homology of the punctured affine line $\mathbb{G}_{m,k}^{\dagger}$ has a 1-cycle given by a system $\epsilon \in \mathbb{Q}_k^{\vee}$ of roots of unity, which is the intuitive "loop" around the origin $0 \in \mathbb{A}_k^{1\dagger}$, while the underlying topological space of $\mathbb{G}_{m,k}^{\dagger}$ itself is contractible. The analytic homologies surely reflects the informations of the analytic structures of an analytic space and a dagger space.

Now the annuli $\mathrm{A}_k^1(1,1)^{\dagger}$ and $\mathbb{G}_{m,k}^{\dagger}$ are analytically pathwise connected as we saw above. Moreover, similar calculations guarantee that the singular homology groups of the annuli $\mathrm{D}_k^{1\dagger} \backslash \{0\}$ and $\mathring{\mathrm{D}}_k^{1\dagger} \backslash \{0\}$ are isomorphic to those of $\mathrm{A}_k^1(1,1)^{\dagger}$ and $\mathbb{G}_{m,k}^{\dagger}$. However, recall that a unit disc $\mathrm{D}_k^{1\dagger} \subset \mathbb{A}_k^{1\dagger}$ will split into infinitely many or two connected components if one removes a point of type 2 or 3 (see [BER1]). Similarly, a general punctured disc with two or more removed points is not analytically pathwise connected. Imagine that removing two distinct k-rational points from $\mathrm{D}_k^{1\dagger}$ causes the partition of $\mathrm{D}_k^{1\dagger}$ corresponding to removing the point of type 2 which is the common summit of the greatest disjoint open discs centred at the removed two k-rational points. As we mentioned in §3.1, the homotopy set $\pi_0(\mathrm{D}_k^{1\dagger} \backslash \{0,1\})$ of the punctured unit disc $\mathrm{D}_k^{1\dagger} \backslash \{0,1\}$ is canonically bijective to the topological homotopy set $\pi_0^{\mathrm{top}}(\mathrm{D}_k^{1\dagger} \backslash \{0,1,x\})$, where $x \in \mathrm{D}_k^{1\dagger} \backslash \{0,1\} \subset \mathrm{D}_k^{1\dagger}$ is the point corresponding to the norm of $k\{T_1\}^{\dagger}$. The proof of the following proposition may help one understand what occurs.

Lemma 5.1.5. *Take an integer $m \in \mathbb{N}$ and m distinct k-rational points $a_1, \ldots, a_m \in D_k^{1\dagger}(k)$. Set*

$$Y \coloneqq D_k^{1\dagger} \backslash \{p(|a_i - a_j|, a_i) \mid i, j = 1, \ldots, m\} \subset D_k^{1\dagger} \backslash \{a_1, \ldots, a_m\},$$

where for each $d \in [0,1]$ and $a \in k$, $p(d,a) \in D_k^{1\dagger}$ is the point corresponding to the norm of $k\{d^{-1}(T_1 - a)\}^{\dagger}$ if $d > 0$ or corresponding to the pull-back seminorm of the norm of k by the bounded k-homomorphism $k\{T_1\}^{\dagger} \to k \colon T \mapsto a$ if $d = 0$. The dagger space Y is locally analytically pathwise connected.

Proof. Trivial because Y is a disjoint union of polydiscs and annuli by the argument in §4.2 in [BER1]. ◻

Note that the numeber of the polydiscs would be infinite if we removed the condition that k is a local field, but that of the annuli would be always finite.

Proposition 5.1.6. *In the situation above, one has a canonical $\mathbb{Z}[G_k]$-equivariant isomorphism*

$$H_0(D_k^{1\dagger} \backslash \{a_1, \ldots, a_m\}) \cong_{\mathbb{Z}} \mathbb{Z}^{\oplus \pi_0^{top}(Y)}$$

and an exact sequence

$$0 \to \hat{\mathbb{Z}}(1)^{\oplus\alpha} \to H_1(D_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}) \to (1 + k^{\circ\circ})^{\oplus\alpha} \to 0$$

of $\mathbb{Z}[G_k]$*-modules, where the action of* G_k *on* $\mathbb{Z}$ *is trivial, and where* $\alpha \in \mathbb{N}$ *is the number of (analytically pathwise) connected components of* Y *which are annuli.*

Proof. The open subspace $Y \subset \mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}$ is the disjoint union of open discs and finitely many open annuli, which are analytically pathwise connected as we saw above. Let Λ (or Λ') be the collection of analytically pathwise connected components of Y which are annuli (resp. discs). Each annulus $D \in \Lambda$ is presented as $\mathring{\mathrm{A}}1_k(|a_j - a_i|, |a_k - a_i|; a_i)^\dagger$ for some $i, j, k = 1, \ldots, m$ such that $|a_j - a_i| < |a_k - a_i|$ and $|a_h - a_i| \notin (|a_j - a_i|, |a_k - a_i|)$ for any $h = 1, \ldots, m$, and each disc $D \in \Lambda'$ is presented as $a + \mathring{\mathrm{D}}_k^1(R_a)^\dagger$ for some $a \in \mathrm{D}_k^{1\dagger}(k)\backslash\{a_1, \ldots, a_m\}$ such that there exist some integers $i, j = 1, \ldots, m$ with $|a_i - a| = |a_j - a| = |a_i - a_j| = R_a$, where $R_a \in (0, \infty)$ is the distance $\min\{|a_i - a| \mid i = 1, \ldots, m\}$ between a and $\{a_1, \ldots, a_m\}$. One has

$$\begin{array}{rcl}
\mathrm{H}_0(Y) & \cong_{\mathbb{Z}[G_k]} & \displaystyle\bigoplus_{D \in \pi_0(Y)} \mathrm{H}_0(D) \cong_{\mathbb{Z}[G_k]} \mathbb{Z}^{\oplus\pi_0(Y)} \cong_{\mathbb{Z}[G_k]} \mathbb{Z}^{\oplus\pi_0^{\mathrm{top}}(Y)} \\
\mathrm{H}_1(Y) & \cong_{\mathbb{Z}[G_k]} & \displaystyle\bigoplus_{D \in \pi_0(Y)} \mathrm{H}_1(D) \cong_{\mathbb{Z}[G_k]} \bigoplus_{D \in \Lambda} \mathrm{H}_1(D) \cong_{\mathbb{Z}[G_k]} \mathrm{H}_1(\mathring{\mathrm{D}}_k^{1\dagger}\backslash\{0\})^{\oplus\alpha},
\end{array}$$

and hence it suffices to show that the canonical $\mathbb{Z}[G_k]$-module homomorphism

$$\begin{array}{rcl}
\pi_0(Y) & \to & \pi_0(\mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}) \\
\mathrm{H}_0(Y) & \to & \mathrm{H}_0(\mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}) \\
\mathrm{C}_1(Y) & \to & \mathrm{C}_1(\mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}) \\
\mathrm{H}_1(Y) & \to & \mathrm{H}_1(\mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\})
\end{array}$$

induced by the open immersion $Y \to \mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}$ are isomorphisms. Therefore we have only to show that any morohism $[0, q_k - 1] \to \mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}$ factors through $Y \to \mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\}$.

Now one has a group-theoretical isomorphism

$$\begin{array}{rcl}
[\tilde{k}^\times] \times \mathbb{Z} \times E_{k,1} \times (1 + (k^\dagger_{[0,q_k-1]})^{\circ\circ}) & \to & \cong (k^\dagger_{[0,q_k-1]})^\times \\
(a, l, x, 1 + g) & \mapsto & a\pi_k^l x(t_1)(1 + g(t_1))
\end{array}$$

in Corollary 2.5.11, where $\pi_k \in k$ is a uniformiser. Take an analytic path $\gamma(t_1) \colon [0, q_k - 1] \to X$. By an ordinary argument, γ is given by a bounded k-homomorphism $f \colon k\{T_1\}^\dagger \to k^\dagger_{[0,q_k-1]}$ such that $f(T_1 - a_1), \ldots, f(T_1 - a_m)$ is invertible in $k^\dagger_{[0,q_k-1]}$. Set $f(T_1) - a_h = a^{(h)}\pi_k^{l_h} x_h(t_1)(1 + g_h(t_1))$ for each $h = 1, \ldots, m$. Since $Y(k) = \mathrm{D}_k^{1\dagger}(k)\backslash\{a_1, \ldots, a_m\} = (\mathrm{D}_k^{1\dagger}\backslash\{a_1, \ldots, a_m\})(k)$, one has $\gamma(0) \in Y(k)$. Let $D \in \Lambda \sqcup \Lambda'$ be the analytically pathwise connected component of $\gamma(0)$ in Y. Suppose $D \in \Lambda$ and D is presented as $a_i + \mathring{\mathrm{A}}_k^1(|a_j - a_i|, |a_k - a_i|)^\dagger$ by some $i, j, k = 1, \ldots, m$ such that $|a_j - a_i| < |a_k - a_i|$ and

$|a_h - a_i| \notin (|a_j - a_i|, |a_k - a_i|)$ for any $h = 1, \ldots, m$. The condition $\gamma(0) \in D$ implies that $|a^{(h)}||\pi_k|^{l_h} \in (|a_j - a_i|, |a_k - a_i|)$ for any $h = 1, \ldots, m$. For any $x \neq 1, x_k \in E_{k,1}$, one has

$$\begin{aligned} & |(f(T_1) - a_i)_x|\|x\| = |(f(T_1) - a_k)_x|\|x\| = |a^{(k)}||\pi_k|^{l_k}\|x\||(g_k)_{x_k^{-1}x}| \\ \leq \ & |a^{(k)}||\pi_k|^{l_k}\|x_k\||(g_k)_{x_k^{-1}x}|\|x_k^{-1}x\| < |a^{(k)}||\pi_k|^{l_k}\|x_k\| = |a^{(i)}||\pi_k|^{l_i}\|x_k\||(g_i)_{x_i^{-1}x_k}| \\ \leq \ & |a^{(i)}||\pi_k|^{l_i}\|x_i\||(g_i)_{x_i^{-1}x_k}|\|x_i^{-1}x_k\| \leq |a^{(i)}||\pi_k|^{l_i}\|x_i\| = |(f(T_1) - a_i)_{x_i}|\|x_i\| \end{aligned}$$

and hence $x_i = x_k$ or $x_i = 1$. Assume $x_i = x_k \neq 1$. One obtains

$$\begin{aligned} & |a^{(i)}||\pi_k|^{l_i}|(g_i)_{x_i^{-1}}| \leq |a^{(i)}||\pi_k|^{l_i}|(g_i)_{x_i^{-1}}|\|x_i\| < |a^{(i)}||\pi_k|^{l_i} < |a_k - a_i| \\ = \ & |(a^{(k)}\pi_k^{l_k}(g_k)_{x_k^{-1}}) + (a_k - a_i)| = |((f(T_1) - a_k) + (a_k - a_i))_1| = |(f(T_1) - a_i)_1| \\ = \ & |a^{(i)}||\pi_k|^{l_i}|(g_i)_{x_i^{-1}}| \end{aligned}$$

and it is a contradiction. Therefore $x_i = 1$. Then one derives

$$\begin{aligned} \|f(T_1) - a_i\| &= |a^{(i)}||\pi_k|^{(q_k-1)l_i} \in (|a_j - a_i|, |a_k - a_i|) \\ \|(f(T_1) - a_i)^{-1}\| &= |a^{(i)}|^{-1}|\pi_k|^{(q_k-1)-l_i} \in (|a_k - a_i|^{-1}, |a_j - a_i|^{-1}) \end{aligned}$$

and hence $\gamma^\sharp([0, q_k - 1]_k) \subset \mathscr{M}(k\{|a_k - a_i|^{-1}(T - a_i), |a_j - a_i|(T - a_i)^{-1}\}) = D$. On the other hand, suppose $D \in \Lambda'$ and D is presented as $a + \mathring{\mathrm{D}}_k^1(R_a)^\dagger$ for some $a \in \mathrm{D}_k^{1\dagger}(k) \backslash \{a_1, \ldots, a_m\}$ such that there exist some integers $i, j = 1, \ldots, m$ with $|a_i - a| = |a_j - a| = |a_i - a_j| = R_a$. The condition $\gamma(0) \in D$ implies that $\|f(T_1) - a\| < R_a = \min\{|a - a_h| \mid h = 1, \ldots, m\}$, and hence $|a^{(h)}||\pi_k|^{l_h} = \|f(T_1) - a_h\| = |(f(T_1) - a) - (a_h - a)| = |a - a_h|$ for any $h = 1, \ldots, m$. For any $x \neq 1, x_j \in E_{k,1}$, one has

$$\begin{aligned} & |(f(T_1) - a_i)_x|\|x\| = |(f(T_1) - a_j)_x|\|x\| = |a^{(j)}||\pi_k|^{l_j}\|x\||(g_j)_{x_j^{-1}x}| \\ \leq \ & |a^{(j)}||\pi_k|^{l_j}\|x_j\||(g_j)_{x_j^{-1}x}|\|x_j^{-1}x\| < |a^{(j)}||\pi_k|^{l_j}\|x_j\| = |a^{(i)}||\pi_k|^{l_i}\|x_j\||(g_i)_{x_i^{-1}x_j}| \\ \leq \ & |a^{(i)}||\pi_k|^{l_i}\|x_i\||(g_i)_{x_i^{-1}x_j}|\|x_i^{-1}x_j\| \leq |a^{(i)}||\pi_k|^{l_i}\|x_i\| = |(f(T_1) - a_i)_{x_i}|\|x_i\| \end{aligned}$$

and hence $x_i = x_j$ or $x_i = 1$. Assume $x_i = x_j \neq 1$. One obtains

$$\begin{aligned} & |a^{(i)}||\pi_k|^{l_i}|(g_i)_{x_i^{-1}}| \leq |a^{(i)}||\pi_k|^{l_i}|(g_i)_{x_i^{-1}}|\|x_i\| < |a^{(i)}||\pi_k|^{l_i} = |a_i - a| \\ = \ & |(a^{(j)}\underline{p}(l_j)(g_j)_{x_j^{-1}}) - (a_i - a_j)| = |((f(T_1) - a_j) - (a_i - a_j))_1| = |(f(T_1) - a_i)_1| \\ = \ & |a^{(i)}||\pi_k|^{l_i}|(g_i)_{x_i^{-1}}| \end{aligned}$$

and it is a contradiction. Therefore $x_i = 1$. Then one derives

$$\begin{aligned} \|f(T_1) - a_i\| &= |a^{(i)}||\pi_k|^{l_i} = |a_i - a| = R_a \\ \|f(T_1) - a\| &= \|(f(T_1) - a_i) + (a_i - a)\| \leq R_a \end{aligned}$$

and hence $\gamma^\sharp([0, q_k - 1]_k) \subset \mathscr{M}(k\{R_a^{-1}(T - a)\}) = D$. We conclude that the morphism γ factors through the open immersion $Y \to \mathrm{D}_k^{1\dagger} \backslash \{a_1, \ldots, a_m\}$, and hence we have done. $\square$

Corollary 5.1.7. *In the situation above, one has a canonical isomorphism*

$$H_0(D_k^{1\dagger}\backslash\{a_1,\ldots,a_m\},\mathbb{Q}_p) \cong_{\mathbb{Q}_p[G_k]} \mathbb{Q}_p^{\oplus\pi_0^{top}(Y)}$$

and a canonical exact sequence

$$0 \to \mathbb{Q}_p(1)^{\oplus\alpha} \to H_1(D_k^{1\dagger}\backslash\{a_1,\ldots,a_m\},\mathbb{Q}_p) \to k^{\oplus\alpha} \to 0$$

of p-adic Galois representations.

Corollary 5.1.8. *In the situation above, one has canonical isomorphisms*

$$H_i(D_k^{1\dagger}\backslash\{a_1,\ldots,a_m\},\mathbb{Q}_l) \cong_{\mathbb{Q}_l[G_k]} \begin{cases} \mathbb{Q}_l^{\oplus\pi_0^{top}(Y)} & (i=0) \\ \mathbb{Q}_l(1)^{\oplus\alpha} & (i=1) \end{cases}$$

of l-adic Galois representations for a prime number $l \neq p \in \mathbb{N}$.

Corollary 5.1.9. *In the situation above, the p-adic analytic homology group $H_i(D_k^{1\dagger}\backslash\{a_1,\ldots,a_m\},\mathbb{Q}_p)$ is a crystalline representation for $i = 0, 1$.*

Next, we see the second example. It is pity that we failed to verify whether the affinoid covering of a Tate curve $\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ given by the two annuli $\mathrm{A}^1(\sqrt{|q|},1)^{\dagger}$ and $\mathrm{A}^1(|q|,\sqrt{|q|})^{\dagger}$ in $\mathbb{G}_{m,k}^{\dagger}$ forms a great covering or not, and hence we can not apply the Mayer-Vietoris exact sequence for its geometrical analytic singular homology. Though they have the universality of great domains in $\mathbb{G}_{m,k}^{\dagger}$, the images in $\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ might not. Therefore we have not determined the homology group of $\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ yet, but we guarantee that a combination of the three specific cycles on $\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ we saw some times is not a boundary.

Proposition 5.1.10. *For an element $q \in k^{\circ\circ}\backslash\{0\}$, one has a canonical $\mathbb{Z}[G_k]$-module isomorphism*

$$H_0(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}) \cong_{\mathbb{Z}} \mathbb{Z},$$

where the action of G_k on $\mathbb{Z}$ on the right hand side is trivial, and a canonical G_k-equivariant embedding

$$L \hookrightarrow H_1(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}})$$

admitting an exact sequence

$$0 \to H_1(\mathbb{G}_{m,k}^{\dagger}) \to L \to \mathbb{Z} \to 0$$

or equivalently

$$0 \to \hat{\mathbb{Z}}(1) \to L \to (1+k^{\circ\circ})q^{\mathbb{Z}} \to 0$$

of $\mathbb{Z}[G_k]$-modules, where the action of G_k on $\mathbb{Z}$ in the right hand side of the first exact sequnece is trivial.

Proof. Since the projection $\mathbb{G}_{m,k}^{\dagger} \to \mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ is surjective, the analytically pathwise connectedness of $\mathbb{G}_{m,k}^{\dagger}$ guarantees that of $\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ and hence $\mathrm{H}_0(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}) \cong_{\mathbb{Z}} \mathbb{Z}$. Let L denote the subgroup of $\mathrm{H}_1(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}})$ generated by the cycles $\gamma_1^{(\epsilon)}, \gamma_2^{(\underline{1+a})}, \gamma_3^{(\underline{q})} \colon [0, q_k - 1] \to \mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ determined by the functions

$$\epsilon(t_1), \left[\underline{1+a}(t_1)\left(-\frac{a}{1-p}\underline{p}(t_1) + \frac{1+a-p}{1-p}\right)^{-1}\right], \underline{q}(t_1)$$

for a system $\epsilon \in E_{k,1}$ of power roots of unity, an element $a \in k^{\circ\circ}$, a system $\underline{1+a} \in E_{k,1}$ of power roots of $1+a$, a system $\underline{p} \in E$ of power roots of p, and a system $\underline{q} \in E_{k,1}$ of power roots of q. For any $\gamma \in L$, present

$$\gamma = b_1\left[\gamma_1^{(\epsilon_1)}\right] + \cdots + b_m\left[\gamma_1^{(\epsilon_m)}\right] + c_1\left[\gamma_2^{(\underline{1+a_1})}\right] + \cdots + c_l\left[\gamma_2^{(\underline{1+a_l})}\right] + d_1\left[\gamma_3^{(\underline{q}_1)}\right] + \cdots + d_k\left[\gamma_3^{(\underline{q}_k)}\right]$$

by integers $m, l, k \in \mathbb{N}$, integers $b_1, \ldots, b_m, c_1, \ldots, c_l, d_1, \ldots, d_k \in \mathbb{Z}$, systems $\epsilon_1, \ldots, \epsilon_m \in E_{k,1}$ of power roots of unity, elements $a_1, \ldots, a_l \in k^{\circ\circ}$, systems $\underline{1+a_1}, \ldots, \underline{1+a_l} \in E_{k,1}$ of power roots of $1+a_1, \ldots, 1+a_l$, and systems $\underline{q}_1, \ldots, \underline{q}_k \in E$ of power roots of q. Since $\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$ is analytically pathwise connected analytic group, one has

$$\gamma = \left[\gamma_1^{(\epsilon_1^{b_1}\cdots\epsilon_m^{b_m})}\right] + \left[\gamma_2^{(\underline{1+a_1}^{c_1}\cdots\underline{1+a_l}^{c_l})}\right] + \left[\gamma_3^{(\underline{q}_1^{d_1}\cdots\underline{q}_k^{d_k})}\right]$$

by Corollary 3.2.24. Now take a system $\underline{q} \in E_{k,1}$ of power roots of q. the function $\underline{q}_1^{d_1} \cdots \underline{q}_k^{d_k}$ is a system ofpower roots of $\underline{q}_1(d_1) \cdots \underline{q}_k(d_k) = q^{d_1+\cdots+d_k} \in k$, and hence the ratio $\underline{q}_1^{d_1} \cdots \underline{q}_k^{d_k}\underline{q}^{-(d_1+\cdots+d_k)}$ is a system of power roots of unity. Since a product and the inverse of systems of power roots of unity are again systems of power roots of unity, and since a product of a system of power roots of unity and a system of power roots of $1+a \in 1+k^{\circ\circ}$ is again a power root system of $1+a$, one derives the presentation

$$\gamma = \left[\gamma_2^{(\underline{1+a})}\right] + d\left[\gamma_3^{(\underline{q})}\right]$$

by an integer $d \in \mathbb{Z}$, an element $a \in \overline{k}^{\circ\circ}$, and a system $\underline{1+a} \in E_{k,1}$ of power roots of $1+a$. We see the uniqueness of this presentation for the fixed system $\underline{q}$. It suffices to show that $\gamma = [\gamma_2^{(\underline{1+a})}] + d[\gamma_3^{(\underline{q})}] \neq 0 \in \mathrm{H}_1(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}})$ for any parametres $(d, a, \underline{1+a}) \neq (0, 0, 1(t_1))$. Take parametres $(d, a, \underline{1+a}) \neq (0, 0, 1(t_1))$, and assume $\gamma = [\gamma_2^{(\underline{1+a})}] + d[\gamma_3^{(\underline{q})}] = 0 \in \mathrm{H}_1(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}})$. Consider the integral of the differential form $T_1^{-1}dT_1 \in \mathrm{H}_0(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}, \Omega_{\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}})$. As we calculated in §4.5, one has

$$\begin{aligned} 0 &= \int_{[0]} \frac{dT_1}{T_1} = \int_{\gamma} \frac{dT_1}{T_1} = (q_k - 1)\log^{\mathrm{r}}(1+a) + (q_k - 1)d\log\underline{q} \\ &= -(q_k - 1)\log(1+a) + (q_k - 1)\log((\underline{1+a})\underline{q}^d). \end{aligned}$$

One obtains $\log((\underline{1+a})\underline{q}^d) = 0$ because $\log(1+a) \in k$, $\log((\underline{1+a})\underline{q}^d) \in \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$, and $k \cap \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}} = 0$. Therefore $(\underline{1+a})\underline{q}^d \in T_{p,k}$ and it follows $(1+a)^{q_k-1}q^{d(q_k-1)} = ((\underline{1+a})\underline{q}^d)(q_k - 1) = 1$. It implies that $d = 0$ and $a = 0$, and it contradicts the assumption $(d, a, \underline{1+a}) \neq (0, 0, 1(t_1))$. Thus one obtains a spritting exact sequence

$$0 \to \mathrm{H}_1(\mathbb{G}_{m,k}^{\dagger}) \to L \to \mathbb{Z} \to 0,$$

where $\mathrm{H}_1(\mathbb{G}_{m,k}^{\dagger}) \to L$ is the homomorphism associated with the canonical projection $\mathbb{G}_{m,k}^{\dagger} \twoheadrightarrow \mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}$, and $L \to \mathbb{Z}$ is the well-defined homomorphism which sends $[\gamma_2^{(\underline{1+a})}] + d[\gamma_3^{(\underline{q})}]$ to d for any parametres $(d, a, \underline{1+a})$. This homomorphism is independent of the choice of $\underline{q}$. Indeed, if one replaces $\underline{q}$ to some other $\underline{q}'$, then one has

$$\left[\gamma_2^{(\underline{1+a})}\right] + d\left[\gamma_3^{(\underline{q})}\right] = d\left[\gamma_1^{(\underline{q}\underline{q}'^{-1})}\right] + \left[\gamma_2^{(\underline{1+a})}\right] + d\left[\gamma_3^{(\underline{q}')}\right] = \left[\gamma_2^{((\underline{1+a})\underline{q}^d\underline{q}'^{-d})}\right] + d\left[\gamma_3^{(\underline{q}')}\right]$$

and the coefficient d does not change. □

Though we have no idea for the calculation of $\mathrm{H}_2(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}})$ now, note that one has

$$\mathrm{H}_2^{\square}(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}}) \neq 0.$$

It contains the non-trivial cycle determined by a function $\epsilon(t_1)\underline{q}(t_2)$ for systems $\epsilon, \underline{q} \in E_{k,1}$ of power roots of $1, q$. The non-vanishing of this cycle in $\mathrm{H}_2^{\square}(\mathbb{G}_{m,k}^{\dagger}/q^{\mathbb{Z}})$ can be verified by the integral of the $(1,1)$-form $dT_1 \wedge d\overline{T_2}$ formally defined by the use of the involution $*\colon k_{[0,q_k-1]^2} \to k_{[0,q_k-1]^2} \colon x(t_1)y(t_2) \mapsto x(t_1)\overline{y}(t_2)$. We do not intend to go too far into the integral of a (p,q)-form because it is useless when we consider the analytic singular homology but not the cubical one. Now finally we see the third example.

Proposition 5.1.11. *One has a canonical $\mathbb{Z}[G_k]$-module isomorphism*

$$H_0(\mathbb{P}_k^{1\dagger}) \cong_{\mathbb{Z}[G_k]} \mathbb{Z},$$

where the action of G_k on $\mathbb{Z}$ in the right hand side is trivial.

Proof. It is trivial because $\mathbb{P}_k^{1\dagger}$ has the non-disconnected covering by two copies of the unit disc $\mathrm{D}_k^{1\dagger}$, which is analytically pathwise connected. □

It is pity that we do not know the covering by the two discs is a universally great covering. If we were to verify it, then the Mayer-Vietoris exact sequence would gurantee that

$$\mathrm{H}_1(\mathbb{P}_k^{1\dagger}/\overline{k}) = 0$$
$$\text{and} \quad \mathrm{H}_2(\mathbb{P}_k^{1\dagger}/\overline{k}) \cong_{\mathbb{Z}} \mathrm{H}_1(\mathrm{A}_k^1(1,1)^{\dagger}/\overline{k}).$$

The non-trivial cycle in $\mathrm{H}_2(\mathbb{P}_k^{1\dagger})$ would be given by the formal combination of the morphisms determined by the functions

$$\left[\epsilon(t_0)\frac{1-\underline{p}(t_1)}{1-p} : 1\right] \text{ and } \left[1 : \epsilon(t_0)\frac{1-\underline{p}(t_1)}{1-p}\right]$$

in the homogeneous coordinate $[z_0 : z_1]$ by systems $\epsilon, \underline{p} \in E_{k,1}$ of power roots of $1, p$.

6 Appendix

6.1 Generalised analytic homologies

We verify the existence of a homotopy between the barycentric subdivision operator and the identity for the "generalised" cubical singular homology. We defined the ring $k_{[0,1]^n}$ of analytic functions on $[0,1]^n$ using a formal symbol $x(a_1 t_1 + \cdots + a_n t_n + c)$ for a character $x \in \mathbb{Q}_k^\vee$ and integers $a_1, \ldots, a_n, c \in \mathbb{Z}$, and the pull-back is valid only by an affine map. Now there are many ways to justify the pull-back by a coordinate change by polynomials; redefine the ring of analytic functions on $[0,1]$ as the completion of the k-algebra generated by the $\mathbb{Z}$-module of formal symbols of the form $x(t^n)$ for $x \in \mathbb{Q}_k^\vee$ and $n \in \mathbb{N}_+$ with respect to an apprpriate norm, for example. Only in this section, we replace the rings of analytic functions on $[0,1]^n$ for each $n \in \mathbb{N}$ to be appropriate ones so that they admit the pull-back by a coordinate change by polynomials. The chain complex $\mathrm{C}_*(X)$ and the cubical singular homology $\mathrm{H}_*(X)$ are also differs from the original ones.

Lemma 6.1.1 (barycentric subdivision for cubes)**.** *Suppose k is algebraically closed. Let X be a k-$\mathscr{A}$ space, $n \in \mathbb{N}$, and $f(t_1, \ldots, t_n) \colon [0,1]^n \to X$ a morphism. Define $Bf \in C_n(X)$ as*

$$Bf := \sum_{I \in \{0,1\}^n} \left[f\left(\frac{t_1 + I_1}{2}, \ldots, \frac{t_n + I_n}{2} \right) \right].$$

It induces a group homomophism

$$\begin{array}{rccc} B \colon & C_n(X) & \to & C_n(X) \\ & [f] & \mapsto & B[f] := Bf \end{array}$$

identifying $C_n(X)$ as the free $\mathbb{Z}$-module generated by non-degenerate paths. Then one has $\overline{\xi} = \overline{B\xi} \in H_n(X)$ for any $\xi \in \ker d_n$.

Proof. It suffices to construct a homotopy $\Phi \colon \mathrm{C}_*(X) \to \mathrm{C}_{*+1}(X)$ connecting $\mathrm{id}, B \in \prod_{n \in \mathbb{N}} \mathrm{End}_{\mathbb{Z}} \mathrm{C}_n(X)$. Set

$$a = (a_n)_{n \in \mathbb{N}} := \left(\sum_{i=1}^{n} \frac{2^{n-i}(2^i + 1)n!}{i!} \right)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}.$$

Define a_n polynomial maps $a_1^{(n)}, \ldots, a_{a_n}^{(n)} \colon [0,1]^{n+1} \to [0,1]^n$ inductively on n in the following way: When $n = 0$, then $a_n = 0$ and hence there is nothing to do. Formally define $a_1^{(0)} = a_2^{(0)}$ as the zero-dimensional collumn vector 0. When $n > 0$, suppose $a_1^{(n-1)}, \ldots, a_{a_{n-1}}^{(n-1)}$ have been defined. For $j = 1$, set

$$\begin{aligned} a_j^{(n)}(t_1, \ldots, t_{n+1}) &:= \left(\frac{(1 - t_{n+1}) + 2t_1 t_{n+1}}{2}, \ldots, \frac{(1 - t_{n+1}) + 2t_n t_{n+1}}{2} \right) \\ &= \left(\frac{1 + (2t_1 - 1)t_{n+1}}{2}, \ldots, \frac{1 + (2t_n - 1)t_{n+1}}{2} \right). \end{aligned}$$

For $j = 2, \dots, 1 + 2^n$, presenting $j = 2 + I_1 + 2I_2 \cdots + 2^{n-1} I_n$ by unique $I = (I_1, \dots, I_n) \in \{0,1\}^n$, set

$$\begin{aligned} a_j^{(n)}(t_1, \dots, t_{n+1}) &:= \left(\frac{(1 - t_{n+1}) + (t_1 + I_1)t_{n+1}}{2}, \dots, \frac{(1 - t_{n+1}) + (t_n + I_1)t_{n+1}}{2} \right) \\ &= \left(\frac{1 + (t_1 + I_1 - 1)t_{n+1}}{2}, \dots, \frac{1 + (t_n + I_n - 1)t_{n+1}}{2} \right). \end{aligned}$$

For $j = 2 + 2^n, \dots, 1 + 2^n + 2na_{n-1} = a_n$, presenting $j = 1 + 2^n + (\sigma n + (i-1))a_{n-1} + h$ by unique $\sigma = 0, 1$, $i = 1, \dots, n$, $h = 1, \dots, a_{n-1}$, and $m = 1, \dots, n$, set

$$\begin{aligned} a_j^{(n)}(t_1, \dots, t_{n+1})_m &:= \begin{cases} 2^{-1}(1 - t_{n+1}) + a_h^{(n-1)}(t_1, \dots, t_n)_m t_{n+1} & (m < i) \\ 2^{-1}(1 + (-1)^\sigma t_{n+1}) & (m = i) \\ 2^{-1}(1 - t_{n+1}) + a_h^{(n-1)}(t_1, \dots, t_n)_{m-1} t_{n+1} & (m > i) \end{cases} \\ &= \begin{cases} 2^{-1}(1 + (2a_h^{(n-1)}(t_1, \dots, t_n)_m - 1)t_{n+1}) & (m < i) \\ 2^{-1}(1 + (-1)^\sigma t_{n+1}) & (m = i) \\ 2^{-1}(1 + (2a_h^{(n-1)}(t_1, \dots, t_n)_{m-1} - 1)t_{n+1}) & (m > i) \end{cases} \end{aligned}$$

and

$$a_j^{(n)}(t_1, \dots, t_{n+1}) = \left(a_j^{(n)}(t_1, \dots, t_{n+1})_m \right)_{m=1}^n .$$

For any sequence $b = (b_0, b_1, \dots)$ of integers of finite length, denote by $\mathrm{tr}(b)$ the total sum $b_0 + b_1 + \cdots$. For each $j = 1, \dots, a_n$, present

$$j = \sum_{l=0}^{m-1} \left(1 + 2^{n-l} + (\sigma_l(n-l) + (i_l - 1))a_{n-l-1} \right) + h$$

by unique $m \in \{0, \dots, n-1\}$, $\sigma_0, \dots, \sigma_{m-1} \in \{0, 1\}$, $(1, \dots, 1) \leq (i_0, \dots, i_{m-1}) \leq (n, n-1, \dots, n-m+1)$, and $h \in \{1, \dots, 1 + 2^{n-m}\}$. Define $\chi_n(j) \in \mathbb{N}$ as

$$\chi_n(j) := \begin{cases} 1 + \mathrm{tr}(i) + \mathrm{tr}(\sigma) & (h = 1) \\ \mathrm{tr}(i) + \mathrm{tr}(\sigma) & (h > 1) \end{cases} .$$

Identifying χ_n as the set-theoretical map $\mathbb{Z}/a_n\mathbb{Z} \cong \{1, \dots, a_n\} \xrightarrow{\chi_n} \mathbb{N}$, extend χ_n on $\mathbb{Z}$ composing the canonical projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/a_n\mathbb{Z}$. For each $n \in \mathbb{N}$ and $f \in [0,1]^n \to X$, set

$$\Phi_n f := \sum_{j=1}^{a_n} (-1)^{\chi_n(j)} \left[(f \circ a_j^{(n)})(t_1, \dots, t_{n+1}) \right] \in \mathrm{C}_{n+1}(X).$$

It induces a group homomorphism

$$\begin{aligned} \Phi_n \colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(X) \\ [f] &\mapsto \Phi_n[f] := \Phi_n f \end{aligned}$$

identifying $\mathrm{C}_n(X)$ as the free $\mathbb{Z}$-module generated by non-degenerate analytic paths. Formally set $\Phi_{-1} \coloneqq 0 \colon 0 \to \mathrm{C}_0(X)$. We verify that Φ is a homotopy connecting id and B, i.e. one has the equality

$$d_{n+1} \circ \Phi_n - \Phi_{n-1} \circ d_n = \mathrm{id} - B \colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$$

for each $n \in \mathbb{N}$. From now on, we often denote the valuables such as $(t_1, \ldots, t_n)$ by the corresponding the collumn vector ${}^{\mathrm{t}}(t_1, \ldots, t_n)$. To begin with, we prepare some facts about $a^{(n)}$. Take an arbitrary $i' = 1, \ldots, n+1$. We consider the integral orthonormal affine map $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n)$ for $\sigma' = 0, 1$. Suppose $i' = n+1$ first. For $j = 1$, one has

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 0 \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix}, \quad a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 1 \end{pmatrix} = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}.$$

For $j = 2, \ldots, 1 + 2^n$, presenting $j = 2 + I_1 + 2I_2 + \cdots + 2^{n-1} I_n$ by the unique $(I_1, \ldots, I_n) \in \{0, 1\}^n$, one has

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 0 \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix}, \quad a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 1 \end{pmatrix} = \begin{pmatrix} 2^{-1}(t_1 + I_1) \\ \vdots \\ 2^{-1}(t_n + I_n) \end{pmatrix},$$

For $j = 2 + 2^n, \ldots, a_n$, presenting $j = 1 + 2^n + \sigma n a_{n-1} + (i-1) a_{n-1} + h$ by unique $\sigma = 0, 1$, $i = 1, \ldots, n$, and $h = 1, \ldots, a_{n-1}$, one has

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 0 \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix}, \quad a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 1 \end{pmatrix} = \begin{pmatrix} a_h^{(n-1)}(t_1, \ldots, t_n)_1 \\ \vdots \\ a_h^{(n-1)}(t_1, \ldots, t_n)_{i-1} \\ 2^{-1}(1 + (-1)^{\sigma}) \\ a_h^{(n-1)}(t_1, \ldots, t_n)_i \\ \vdots \\ a_h^{(n-1)}(t_1, \ldots, t_n)_{n-1} \end{pmatrix} = \begin{pmatrix} a_h^{(n-1)}(t_1, \ldots, t_n)_1 \\ \vdots \\ a_h^{(n-1)}(t_1, \ldots, t_n)_i \\ 2^{-1}(1 - \sigma) \\ a_h^{(n-1)}(t_1, \ldots, t_n)_{i+1} \\ \vdots \\ a_h^{(n-1)}(t_1, \ldots, t_n)_n \end{pmatrix}.$$

Suppose $i' = n$ next. For $j = 1$, one has

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 0 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 + (2t_1 - 1)t_n) \\ \vdots \\ 2^{-1}(1 + (2t_{n-1} - 1)t_n) \\ 2^{-1}(1 - t_n) \end{pmatrix}, \quad a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 1 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 + (2t_1 - 1)t_n) \\ \vdots \\ 2^{-1}(1 + (2t_{n-1} - 1)t_n) \\ 2^{-1}(1 + t_n) \end{pmatrix}.$$

For $j = 2, \dots, 1 + 2^n$, presenting $j = 2 + I_1 + 2I_2 + \cdots + 2^{n-1} I_n$ by the unique $(I_1, \dots, I_n) \in \{0, 1\}^n$, one has

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 0 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 + (t_1 + I_1 - 1)t_n) \\ \vdots \\ 2^{-1}(1 + (t_{n-1} + I_{n-1} - 1)t_n) \\ 2^{-1}(1 - (1 - I_n)t_n) \end{pmatrix}, \quad a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 1 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 + (t_1 + I_1 - 1)t_n) \\ \vdots \\ 2^{-1}(1 + (t_{n-1} + I_{n-1} - 1)t_n) \\ 2^{-1}(1 + I_n t_n) \end{pmatrix}.$$

For $j = 2 + 2^n, \dots, a_n$, present $j = 1 + 2^n + \sigma n a_{n-1} + (i-1)a_{n-1} + h$ by unique $\sigma = 0, 1$, $i = 1, \dots, n$, and $h = 1, \dots, a_{n-1}$. If $h = 1$, one has

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 0 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 - t_n) + 2^{-1}t_n \\ \vdots \\ 2^{-1}(1 - t_n) + 2^{-1}t_n \\ 2^{-1}(1 + (-1)^\sigma t_n) \\ 2^{-1}(1 - t_n) + 2^{-1}t_n \\ \vdots \\ 2^{-1}(1 - t_n) + 2^{-1}t_n \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \\ 2^{-1}(1 + (-1)^\sigma t_n) \\ 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix}$$

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 1 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 - t_n) + t_1 t_n \\ \vdots \\ 2^{-1}(1 - t_n) + t_{i-1}t_n \\ 2^{-1}(1 + (-1)^\sigma t_n) \\ 2^{-1}(1 - t_n) + t_i t_n \\ \vdots \\ 2^{-1}(1 - t_n) + t_{n-1}t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 + (2t_1 - 1)t_n) \\ \vdots \\ 2^{-1}(1 + (2t_{i-1} - 1)t_n) \\ 2^{-1}(1 + (-1)^\sigma t_n) \\ 2^{-1}(1 + (2t_i - 1)t_n) \\ \vdots \\ 2^{-1}(1 + (2t_{n-1} - 1)t_n) \end{pmatrix},$$

if $h = 2, \dots, 1 + 2^{n-1}$, presenting $h = 1 + I_1 + 2I_2 + \cdots + 2^{n-2} I_{n-1}$ by unique $(I_1, \dots, I_{n-1}) \in \{0, 1\}^{n-1}$, one has

$$a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 0 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1 - t_n) + 2^{-1}t_n \\ \vdots \\ 2^{-1}(1 - t_n) + 2^{-1}t_n \\ 2^{-1}(1 + (-1)^\sigma t_n) \\ 2^{-1}(1 - t_n) + 2^{-1}t_n \\ \vdots \\ 2^{-1}(1 - t_n) + 2^{-1}t_n \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \\ 2^{-1}(1 + (-1)^\sigma t_n) \\ 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix}$$

$$a_j^{(n)}\begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 1 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1-t_n)+2^{-1}(t_1+I_1)t_n \\ \vdots \\ 2^{-1}(1-t_n)+2^{-1}(t_{i-1}+I_{i-1})t_n \\ 2^{-1}(1+(-1)^{\sigma}t_n) \\ 2^{-1}(1-t_n)+2^{-1}(t_i+I_i)t_n \\ \vdots \\ 2^{-1}(1-t_n)+2^{-1}(t_{n-1}+I_{n-1})t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1+(t_1+I_1-1)t_n) \\ \vdots \\ 2^{-1}(1+(t_{i-1}+I_{i-1}-1)t_n) \\ 2^{-1}(1+(-1)^{\sigma}t_n) \\ 2^{-1}(1+(t_i+I_i-1)t_n) \\ \vdots \\ 2^{-1}(1+(t_{n-1}+I_{n-1}-1)t_n) \end{pmatrix},$$

and if $h = 2+2^{n-1}, \ldots, 1+2^{n-1}+2(n-1)a_{n-2} = a_{n-1}$, presenting $h = 1+2^{n-1}+(\sigma_1(n-1)+(i_1-1))a_{n-2}+h_1$ by unique $\sigma_1 = 0, 1$, $i_1 = 1, \ldots, n-1$, and $h_1 = 1, \ldots, a_{n-2}$, one has

$$a_j^{(n)}\begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 0 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1-t_n)+2^{-1}t_n \\ \vdots \\ 2^{-1}(1-t_n)+2^{-1}t_n \\ 2^{-1}(1+(-1)^{\sigma}t_n) \\ 2^{-1}(1-t_n)+2^{-1}t_n \\ \vdots \\ 2^{-1}(1-t_n)+2^{-1}t_n \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \\ 2^{-1}(1+(-1)^{\sigma}t_n) \\ 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix}$$

$$a_j^{(n)}\begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 1 \\ t_n \end{pmatrix} = \begin{pmatrix} 2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\ldots,t_{n-1})_1 t_n \\ \vdots \\ 2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\ldots,t_{n-1})_{i_1} t_n \\ 2^{-1}(1-t_n)+2^{-1}(1+(-1)^{\sigma_1}t_{n-1})t_n \\ 2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\ldots,t_{n-1})_{i_1+1} t_n \\ \vdots \\ 2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\ldots,t_{n-1})_{i-2} t_n \\ 2^{-1}(1+(-1)^{\sigma}t_n) \\ 2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\ldots,t_{n-1})_{i-1} t_n \\ \vdots \\ 2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\ldots,t_{n-1})_{n-2} t_n \end{pmatrix} \quad (i_1 < i)$$

$$
\begin{aligned}
&= \begin{pmatrix}
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_1 t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i_1} t_n \\
2^{-1}(1+(-1)^{\sigma_1} t_{n-1} t_n) \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i_1+1} t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i-1} t_n \\
2^{-1}(1+(-1)^{\sigma} t_n) \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i} t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{n-2} t_n
\end{pmatrix} \\
a_j^{(n)} \begin{pmatrix} t_1 \\ \vdots \\ t_{n-1} \\ 1 \\ t_n \end{pmatrix}
&= \begin{pmatrix}
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_1 t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i} t_n \\
2^{-1}(1+(-1)^{\sigma} t_n) \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i+1} t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i_1} t_n \\
2^{-1}(1-t_n)+2^{-1}(1+(-1)^{\sigma_1} t_{n-1}) t_n \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i_1+1} t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{n-2} t_n
\end{pmatrix} \quad (i_1 \geq i) \\
&= \begin{pmatrix}
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_1 t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i_1} t_n \\
2^{-1}(1+(-1)^{\sigma_1} t_{n-1} t_n) \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i_1+1} t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i-2} t_n \\
2^{-1}(1+(-1)^{\sigma} t_n) \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{i-1} t_n \\
\vdots \\
2^{-1}(1-t_n)+a_{h_1}^{(n-2)}(t_1,\dots,t_{n-1})_{n-2} t_n
\end{pmatrix}.
\end{aligned}
$$

Now set

$$
\begin{aligned}
c_{m'}(\sigma_0,\dots,\sigma_{m'}) &\coloneqq \begin{pmatrix} 2^{-1}(1+(-1)^{\sigma_0}t_n) \\ 2^{-1}(1-t_n)+2^{-1}(1+(-1)^{\sigma_1}t_{n-1})t_n \\ \vdots \\ \sum_{l=0}^{m'}\left(2^{-1}(1-t_{n-l})\prod_{l'=0}^{l-1}t_{n-l'}\right)+2^{-1}(1+(-1)^{\sigma_{m'}}t_{n-m'})\prod_{l'=0}^{m'-1}t_{n-l'} \end{pmatrix} \\
&= \begin{pmatrix} 2^{-1}(1+(-1)^{\sigma_0}t_n) \\ 2^{-1}(1+(-1)^{\sigma_1}t_{n-1}t_n) \\ \vdots \\ 2^{-1}\left(1+(-1)^{\sigma_{m'}}\prod_{l'=n-m'}^{n}t_{l'}\right) \end{pmatrix} \in \mathbb{Z}[2^{-1},t_{n-m'},\dots,t_n]^{m'+1}
\end{aligned}
$$

for each $m'=1,\dots,n-1$ and $\sigma_0,\dots,\sigma_{m'}\in\{0,1\}^{m'+1}$. In particular one has

$$
c_{m'}(\sigma_0,\dots,\sigma_{m'})_l = c_{m'-1}(\sigma_0,\dots,\sigma_{m'-1})_l
$$

for any $m'=0,\dots,n-1$, $\sigma_0,\dots,\sigma_{m'}\in\{0,1\}$, and $l=1,\dots,m'$;

$$
\begin{aligned}
c_{n-1}(\sigma_0,\dots,\sigma_{n-1}) &= \begin{pmatrix} 2^{-1}(1+(-1)^{\sigma_0}t_n) \\ 2^{-1}(1+(-1)^{\sigma_1}t_{n-1}t_n) \\ \vdots \\ 2^{-1}\left(1+(-1)^{\sigma_{n-1}}\prod_{l'=1}^{n}t_{l'}\right) \end{pmatrix} \\
&= a^{(n)}_{1+2^n+\sigma_0 n a_{n-1}+\sigma_1(n-1)a_{n-2}+\dots+\sigma_{n-2}a_2+1}(1-\sigma_{n-1},t_1,\dots,t_n)
\end{aligned}
$$

for any $\sigma_0,\dots,\sigma_{n-1}\in\{0,1\}$; and

$$
\begin{aligned}
c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'})_{n-i'+1} &= (2^{-1}\left(1+(-1)^{\sigma_{n-i'}}\prod_{l'=i'}^{n}t_{l'}\right) \\
&= a^{(n-i')}_{1+2^{n-i'}+((1-\sigma_{n-i'})(n-i')+i''-1)a_{n-i'-1}+h'}(T(i'))_{i''}
\end{aligned}
$$

for any $i'=1,\dots,n-1$, $i''=1,\dots,n-i'$ and $h'=1,\dots,a_{n-i'-1}$. We will use these equalities without mentioning.

Formally denote by c_{-1} the zero-dimensional collumn vector 0. Set

$$
T(l) = \begin{pmatrix} T(l)_1 \\ \vdots \\ T(l)_l \end{pmatrix} \coloneqq \begin{pmatrix} t_1 \\ \vdots \\ t_{l-1} \\ \prod_{l'=l}^{n} t_{l'} \end{pmatrix}
$$

for each $l=1,\dots,n$, and formally set

$$
T(n+1) = \begin{pmatrix} T(n+1)_1 \\ \vdots \\ T(n+1)_{n+1} \end{pmatrix} \coloneqq \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 1 \end{pmatrix}.
$$

For general $i' = 1, \dots, n+1$ and $j = 1, \dots, a_n$, present

$$j = \sum_{l=0}^{m-1} \left(1 + 2^{n-l} + (\sigma_l(n-l) + (i_l - 1))a_{n-l-1}\right) + h$$

by unique $m \in \{0, \dots, n-1\}$, $\sigma_0, \dots, \sigma_{m-1} \in \{0,1\}$, $(1, \dots, 1) \le (i_0, \dots, i_{m-1}) \le (n, n-1, \dots, n-m+1)$, and $h \in \{1, \dots, 1 + 2^{n-m}\}$. There are five cases. The first one is the case $i' < n-m+1$ and $h = 1$, the second one is the case $i' < n-m+1$ and $1 < h \le 1+2^{n-m}$,the third one is the case $i' = n-m+1$ and $h = 1$, the fourth one is the case $i' = n-m+1$ and $1 < h \le 1 + 2^{n-m}$, and the fifth one is the case $i' > n-m+1$. Let $\sigma' = 0, 1$.

First if $i' < n-m+1$ and $h = 1$, then $a_j^{(n)}(t_1, \dots, t_{i'-1}, \sigma', t_{i'}, \dots, t_n)$ is a collumn vector which can be obtained by rearranging and inserting $c_m(\sigma_0, \dots, \sigma_{m-1}, 1-\sigma')$ in

$$\begin{aligned}
&\begin{pmatrix} \sum_{l=0}^{m} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + t_1 \prod_{l'=0}^{m} t_{n-l'} \\ \vdots \\ \sum_{l=0}^{m} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + t_{n-m-1} \prod_{l'=0}^{m} t_{n-l'} \end{pmatrix} \\
= &\begin{pmatrix} 2^{-1}\left(1 - \prod_{l'=n-m}^{n} t_{l'}\right) + t_1 \prod_{l'=n-m}^{n} t_{l'} \\ \vdots \\ 2^{-1}\left(1 - \prod_{l'=n-m}^{n} t_{l'}\right) + t_{n-m-1} \prod_{l'=n-m}^{n} t_{l'} \end{pmatrix} \\
= &\begin{pmatrix} 2^{-1}\left(1 + (2t_1 - 1) \prod_{l'=n-m}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1 + (2t_{n-m-1} - 1) \prod_{l'=n-m}^{n} t_{l'}\right) \end{pmatrix} = a_1^{(n-m-1)}(T(n-m)).
\end{aligned}$$

Secondly if $i' < n-m+1$ and $1 < h \le 1+2^{n-m}$, then presenting $h = 2 + I_1 + 2I_2 + \cdots + 2^{n-m-1} I_{n-m}$ by unique $(I_1, \dots, I_{n-m}) \in \{0,1\}^{n-m}$, $a_j^{(n)}(t_1, \dots, t_{i'-1}, \sigma', t_{i'}, \dots, t_n)$ is a collumn vector which can be obtained by rearranging and inserting $c_{m-1}(\sigma_0, \dots, \sigma_{m-1})$ in

$$\begin{pmatrix} \sum_{l=0}^{m} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(t_1 + I_1) \prod_{l'=0}^{m} t_{n-l'} \\ \vdots \\ \sum_{l=0}^{m} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(t_{i'-1} + I_{i'-1}) \prod_{l'=0}^{m} t_{n-l'} \\ \sum_{l=0}^{m} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(\sigma' + I_{i'}) \prod_{l'=0}^{m} t_{n-l'} \\ \sum_{l=0}^{m} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(t_{i'} + I_{i'+1}) \prod_{l'=0}^{m} t_{n-l'} \\ \vdots \\ \sum_{l=0}^{m} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(t_{n-m-1} + I_{n-m}) \prod_{l'=0}^{m} t_{n-l'} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-1}\left(1+(t_1+I_1-1)\prod_{l'=n-m}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+(t_{i'-1}+I_{i'-1}-1)\prod_{l'=n-m}^{n} t_{l'}\right) \\ 2^{-1}\left(1+(\sigma'+I_{i'}-1)\prod_{l'=n-m}^{n} t_{l'}\right) \\ 2^{-1}\left(1+(t_{i'}+I_{i'+1}-1)\prod_{l'=n-m}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+(t_{n-m-1}+I_{n-m}-1)\prod_{l'=n-m}^{n} t_{l'}\right) \end{pmatrix}.$$

More concretely, set $(I'_1, \ldots, I'_{n-m-1}) \coloneqq (I_1, \ldots, I_{i'-1}, I_{i'}, \ldots, I_{n-m})$. When $\sigma' = 1 - I_{i'}$, then $a_j^{(n)}(t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n)$ is obtained by rearranging and inserting $c_{m-1}(\sigma_0, \ldots, \sigma_{m-1})$ and 2^{-1} in

$$\begin{pmatrix} 2^{-1}\left(1+(t_1+I_1-1)\prod_{l'=n-m}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+(t_{i'-1}+I_{i'-1}-1)\prod_{l'=n-m}^{n} t_{l'}\right) \\ 2^{-1}\left(1+(t_{i'}+I_{i'+1}-1)\prod_{l'=n-m}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+(t_{n-m-1}+I_{n-m}-1)\prod_{l'=n-m}^{n} t_{l'}\right) \end{pmatrix} = a^{(n-m-1)}_{2+I'_1+2I'_2+\cdots+2^{n-m-2}I'_{n-m-1}}(T(n-m));$$

and when $\sigma' = I_{i'}$, then $a_j^{(n)}(t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n)$ is obtained by rearranging and inserting $c_m(\sigma_0, \ldots, \sigma_{m-1}, 1-\sigma')$ in

$$a^{(n-m-1)}_{2+I'_1+2I'_2+\cdots+2^{n-m-2}I'_{n-m-1}}(T(n-m)).$$

Thirdly if $i' = n-m+1$ and $h = 1$, then $a_j^{(n)}(t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n)$ is a collumn vector which can be obtained by rearranging and inserting $c_{m-1}(\sigma_0, \ldots, \sigma_{m-1})$ in

$$\begin{pmatrix} \sum_{l=0}^{m-1}\left(2^{-1}(1-t_{n-l})\prod_{l'=0}^{l-1} t_{n-l'}\right) + \left(2^{-1}(1-\sigma')\prod_{l'=0}^{m-1} t_{n-l'}\right) + \sigma' t_1 \prod_{l'=0}^{m-1} t_{n-l'} \\ \vdots \\ \sum_{l=0}^{m-1}\left(2^{-1}(1-t_{n-l})\prod_{l'=0}^{l-1} t_{n-l'}\right) + \left(2^{-1}(1-\sigma')\prod_{l'=0}^{m-1} t_{n-l'}\right) + \sigma' t_{n-m} \prod_{l'=0}^{m-1} t_{n-l'} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-1}\left(1-\prod_{l'=n-m+1}^{n} t_{l'}\right) + 2^{-1}(1+\sigma'(2t_1-1))\prod_{l'=n-m+1}^{n} t_{l'} \\ \vdots \\ 2^{-1}\left(1-\prod_{l'=n-m+1}^{n} t_{l'}\right) + 2^{-1}(1+\sigma'(2t_{n-m}-1))\prod_{l'=n-m+1}^{n} t_{l'} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-1}\left(1+\sigma'(2t_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(2t_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \end{pmatrix}.$$

When $\sigma' = 0$, then one has

$$\begin{pmatrix} 2^{-1}\left(1+\sigma'(2t_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(2t_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix},$$

and when $\sigma' = 1$, then one has

$$\begin{aligned}\begin{pmatrix} 2^{-1}\left(1+\sigma'(2t_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(2t_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right)\end{pmatrix} &= \begin{pmatrix} 2^{-1}\left(1+(2t_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+(2t_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right)\end{pmatrix} \\ &= a_1^{(n-m)}(T(n-m+1)).\end{aligned}$$

In particular When $\sigma' = 1$ and $m = 0$, one obtains

$$a_j^{(n)}(t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n) = a_1^{(n)}(t_1, \ldots, t_n, 1) = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$$

again.

Fourthly if $i' = n-m+1$ and $1 < h \le 1+2^{n-m}$, then presenting $h = 2+I_1+2I_2+\cdots+2^{n-m-1}I_{n-m}$ by unique $(I_1, \ldots, I_{n-m}) \in \{0,1\}^{n-m}$, $a_j^{(n)}(t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n)$ is a collumn vector which can be obtained by rearranging and inserting $c_{m-1}(\sigma_0, \ldots, \sigma_{m-1})$ in

$$\begin{aligned}&\begin{pmatrix} \sum_{l=0}^{m-1}\left(2^{-1}(1-t_{n-l})\prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(1-\sigma')\prod_{l'=0}^{m-2} t_{n-l'} + 2^{-1}\sigma'(t_1+I_1)\prod_{l'=0}^{m-1} t_{n-l'} \\ \vdots \\ \sum_{l=0}^{m-1}\left(2^{-1}(1-t_{n-l})\prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(1-\sigma')\prod_{l'=0}^{m-2} t_{n-l'} + 2^{-1}\sigma'(t_{n-m}+I_{n-m})\prod_{l'=0}^{m-1} t_{n-l'}\end{pmatrix} \\ =&\begin{pmatrix} 2^{-1}\left(1-\prod_{l'=n-m+1}^{n} t_{l'}\right) + 2^{-1}(1-\sigma')\prod_{l'=n-m+1}^{n} t_{l'} + 2^{-1}\sigma'(t_1+I_1)\prod_{l'=n-m+1}^{n} t_{l'} \\ \vdots \\ 2^{-1}\left(1-\prod_{l'=n-m+1}^{n} t_{l'}\right) + 2^{-1}(1-\sigma')\prod_{l'=n-m+1}^{n} t_{l'} + 2^{-1}\sigma'(t_{n-m}+I_{n-m})\prod_{l'=n-m+1}^{n} t_{l'}\end{pmatrix} \\ =&\begin{pmatrix} 2^{-1}\left(1+\sigma'(t_1+I_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(t_{n-m}+I_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right)\end{pmatrix}.\end{aligned}$$

When $\sigma' = 0$ then one has

$$\begin{pmatrix} 2^{-1}\left(1+\sigma'(t_1+I_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(t_{n-m}+I_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right)\end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1}\end{pmatrix}$$

and when $\sigma' = 1$, then one has

$$\begin{aligned}\begin{pmatrix} 2^{-1}\left(1+\sigma'(t_1+I_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(t_{n-m}+I_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right)\end{pmatrix} &= \begin{pmatrix} 2^{-1}\left(1+(t_1+I_1-1)\prod_{l'=n-m+1}^{n} t_{l'}\right) \\ \vdots \\ 2^{-1}\left(1+(t_{n-m}+I_{n-m}-1)\prod_{l'=n-m+1}^{n} t_{l'}\right)\end{pmatrix} \\ &= a^{(n-m)}_{2+I_1+2I_2+\cdots+2^{n-m-1}I_{n-m}}(T(n-m+1)).\end{aligned}$$

In particular when $\sigma' = 1$ and $m = 0$, one obtains

$$a_j^{(n)}(t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n) = a^{(n)}_{2+I_1+2I_2+\cdots+2^{n-1}I_n}(t_1, \ldots, t_n, 1) = \begin{pmatrix} 2^{-1}(t_1+I_1) \\ \vdots \\ 2^{-1}(t_n+I_n)\end{pmatrix}$$

again.

Finally if $m + i' > n + 1$, then setting

$$\begin{aligned} h' &:= j - \sum_{m'=0}^{n-i'+1} (1 + 2^{n-m'} + (\sigma_{m'}(n-m') + (i_{m'} - 1))a_{n-m'-1}) \\ &= \sum_{m'=n-i'+2}^{m-1} (1 + 2^{n-m'} + (\sigma_{m'}(n-m') + (i_{m'} - 1))a_{n-m'-1}) + h, \end{aligned}$$

$a_j^{(n)}(t_1, \ldots, t_{i'-1}, \sigma', t_{i'}, \ldots, t_n)$ is a collumn vector which can be obtained by rearranging and inserting $c_{n-i'}(\sigma_0, \ldots, \sigma_{n-i'})$ in

$$\begin{pmatrix} \sum_{l=0}^{n-i'} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_1\right) \prod_{l'=0}^{n-i'} t_{n-l'} \\ \vdots \\ \sum_{l=0}^{n-i'} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i_{n-i'+1}-1}\right) \prod_{l'=0}^{n-i'} t_{n-l'} \\ \sum_{l=0}^{n-i'} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + 2^{-1}(1 + (-1)^{\sigma_{n-i'+1}}\sigma') \prod_{l'=0}^{n-i'} t_{n-l'} \\ \sum_{l=0}^{n-i'} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i_{n-i'+1}}\right) \prod_{l'=0}^{n-i'} t_{n-l'} \\ \vdots \\ \sum_{l=0}^{n-i'} \left(2^{-1}(1-t_{n-l}) \prod_{l'=0}^{l-1} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i'-2}\right) \prod_{l'=0}^{n-i'} t_{n-l'} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-1}\left(1 - \prod_{l'=0}^{n-i'} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_1\right) \prod_{l'=0}^{n-i'} t_{n-l'} \\ \vdots \\ 2^{-1}\left(1 - \prod_{l'=0}^{n-i'} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i_{n-i'+1}-1}\right) \prod_{l'=0}^{n-i'} t_{n-l'} \\ 2^{-1}\left(1 - \prod_{l'=0}^{n-i'} t_{n-l'}\right) + 2^{-1}(1 + (-1)^{\sigma_{n-i'+1}}\sigma') \prod_{l'=0}^{n-i'} t_{n-l'} \\ 2^{-1}\left(1 - \prod_{l'=0}^{n-i'} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i_{n-i'+1}}\right) \prod_{l'=0}^{n-i'} t_{n-l'} \\ \vdots \\ 2^{-1}\left(1 - \prod_{l'=0}^{n-i'} t_{n-l'}\right) + \left(2^{-1}(1-\sigma') + \sigma' a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i'-2}\right) \prod_{l'=0}^{n-i'} t_{n-l'} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-1}\left(1 + \sigma'(2a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_1 - 1) \prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1 + \sigma'(2a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i_{n-i'+1}-1} - 1) \prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1 + \sigma'(-1)^{\sigma_{n-i'+1}} \prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1 + \sigma'(2a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i_{n-i'+1}} - 1) \prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1 + \sigma'(2a_{h'}^{(i'-2)}(t_1, \ldots, t_{i'-1})_{i'-1}) \prod_{l'=0}^{n-i'} t_{n-l'}\right) \end{pmatrix}.$$

The latter collumn vector can be calcluted more concretely. If $\sigma' = 0$, one has

$$\begin{pmatrix} 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_1 - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i_{n-i'+1}-1} - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1+\sigma'(-1)^{\sigma_{n-i'+1}}\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i_{n-i'+1}} - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i'-1})\prod_{l'=0}^{n-i'} t_{n-l'}\right) \end{pmatrix} = \begin{pmatrix} 2^{-1} \\ \vdots \\ 2^{-1} \end{pmatrix},$$

and if $\sigma' = 1$, one has

$$\begin{aligned} &\begin{pmatrix} 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_1 - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i_{n-i'+1}-1} - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1+\sigma'(-1)^{\sigma_{n-i'+1}}\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i_{n-i'+1}} - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1+\sigma'(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i'-1})\prod_{l'=0}^{n-i'} t_{n-l'}\right) \end{pmatrix} \\ = &\begin{pmatrix} 2^{-1}\left(1+(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_1 - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1+(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i_{n-i'+1}-1} - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1+(-1)^{\sigma_{n-i'+1}}\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ 2^{-1}\left(1+(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i_{n-i'+1}} - 1)\prod_{l'=0}^{n-i'} t_{n-l'}\right) \\ \vdots \\ 2^{-1}\left(1+(2a_{h'}^{(i'-2)}(t_1,\dots,t_{i'-1})_{i'-1})\prod_{l'=0}^{n-i'} t_{n-l'}\right) \end{pmatrix} \\ = &\ a^{(i'-1)}_{1+2^{i'-1}+(\sigma_{n-i'+1}(i'-1)+i_{n-i'+1}-1)a_{i'-2}+h'}(T(i')). \end{aligned}$$

In particular when $i' = n+1$, then $a_j^{(n)}(t_1,\dots,t_{i'-1},1,t_{i'},\dots,t_n)$ is the collumn vector $a^{(n)}_{1+2^n+(\sigma_0 n+i_0-1)a_{n-1}+h'}(t_1,\cdots,t_n,1)$ and is obtained by inserting $2^{-1}(1+(-1)^{\sigma_0}) = 1-\sigma_0$ in $a_{h'}^{(n-1)}(t_1,\cdots,t_n)$.

We prepare notaion. These collumn vectors are bothersome to write in equalities. For a collumn vector v, denote its length by $\dim v$. For collumn vectors v_1, v_2 and an injective map $\iota\colon \{1,\dots,\dim v_2\} \hookrightarrow \{1,\dots,\dim v_1+\dim v_2\}$, denote by

$$s\begin{pmatrix} v_1 \\ v_2 \\ \iota \end{pmatrix}$$

the collumn vector obtained by rearranging and inserting v_2 by ι to v_1, and set

$$\mathscr{S}(\iota) := (-1)^{(\iota(1)-1)+\cdots+(\iota(\dim v_2)-\dim v_2)} \in \{1,-1\}.$$

Moreover for a constrant c and $\iota'\colon \{1,\ldots,\dim v_2+1\} \hookrightarrow \{1,\ldots,\dim v_1+\dim v_2+1\}$, set

$$s\begin{pmatrix} v_1 \\ v_2, c \\ \iota' \end{pmatrix} := s\begin{pmatrix} v_1 \\ \begin{pmatrix} v_2 \\ c \end{pmatrix} \\ \iota' \end{pmatrix}.$$

For integers $l \geq m$, denote by

$$\mathscr{I}\begin{pmatrix} l \\ m \end{pmatrix}$$

the set of all injective maps $\iota\colon \{1,\ldots,m\} \hookrightarrow \{1,\ldots,l+m\}$. Set $T_{\sigma'} := (t_1,\ldots,t_{i'-1},\sigma',t_{i'}\ldots,t_n)$. Denote by $S_{n,m} \subset \mathbb{N}^m$ the set of all multiindices $i = (i_0,\ldots,i_{m-1})$ such that $1 \leq i_l \leq n-l$ for any $l = 0,\ldots,m-1$.

Using this result, we complete the calculation for our homotopy. Take an arbitrary analytic path $f\colon [0,1]^n \to X$.

$$\begin{aligned}
(d_{n+1}\circ\Phi_n)f &= d_{n+1}\sum_{h=1}^{a_n}(-1)^{\chi_n(h)}\left[f\circ a_h^{(n)}\right] = \sum_{i'=1}^{n+1}\sum_{\sigma'=0}^{1}\sum_{h=1}^{a_n}(-1)^{i'+\sigma'+\chi_n(h)}\left[f(a_h^{(n)}(T_{\sigma'}))\right] \\
&= \sum_{i'=1}^{n}\sum_{m=0}^{n-i'}\sum_{\sigma'=0}^{1}\sum_{\sigma\in\{0,1\}^m}\sum_{i\in S_{n,m}}(-1)^{i'+\sigma'+1+\mathrm{tr}(i)+\mathrm{tr}(\sigma)}\left[f\left(a^{(n)}_{\sum_{l=0}^{m-1}(1+2^{n-l}+(\sigma_l(n-l)+i_l-1)a_{n-l-1})+1}(T_{\sigma'})\right)\right] \\
&\quad+\sum_{i'=1}^{n}\sum_{m=0}^{n-i'}\sum_{\sigma'=0}^{1}\sum_{\sigma\in\{0,1\}^m}\sum_{i\in S_{n,m}}\sum_{h'=2}^{1+2^{n-m}}(-1)^{i'+\sigma'+\mathrm{tr}(i)+\mathrm{tr}(\sigma)}\left[f\left(a^{(n)}_{\sum_{l=0}^{m-1}(1+2^{n-l}+(\sigma_l(n-l)+i_l-1)a_{n-l-1})+h'}(T_{\sigma'})\right)\right] \\
&\quad+\sum_{i'=2}^{n+1}\sum_{\sigma'=0}^{1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{i\in S_{n,n-i'+1}}(-1)^{i'+\sigma'+1+\mathrm{tr}(i)+\mathrm{tr}(\sigma)}\left[f\left(a^{(n)}_{\sum_{l=0}^{n-i'}(1+2^{n-l}+(\sigma_l(n-l)+i_l-1)a_{n-l-1})+1}(T_{\sigma'})\right)\right] \\
&\quad+\sum_{i'=2}^{n+1}\sum_{\sigma'=0}^{1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{i\in S_{n,n-i'+1}}\sum_{h'=2}^{1+2^{i'-1}}(-1)^{i'+\sigma'+\mathrm{tr}(i)+\mathrm{tr}(\sigma)}\left[f\left(a^{(n)}_{\sum_{l=0}^{n-i'}(1+2^{n-l}+(\sigma_l(n-l)+i_l-1)a_{n-l-1})+h'}(T_{\sigma'})\right)\right] \\
&\quad+\sum_{i'=3}^{n+1}\sum_{\sigma'=0}^{1}\sum_{\sigma\in\{0,1\}^{n-i'+2}}\sum_{i\in S_{n,n-i'+2}}\sum_{h'=1}^{a_{i'-2}}(-1)^{i'+\sigma'+\mathrm{tr}(i)+\mathrm{tr}(\sigma)+\chi_{i'-2}(h')}\left[f\left(a^{(n)}_{\sum_{l=0}^{n-i'+1}(1+2^{n-l}+(\sigma_l(n-l)+i_l-1)a_{n-l-1})+h'}(T_{\sigma'})\right)\right] \\
&= \sum_{m=0}^{n-1}\sum_{\sigma'=0}^{1}\sum_{\sigma\in\{0,1\}^m}\sum_{\iota\in\mathscr{I}_{n-m-1,m+1}}(-1)^{\sigma'+1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_1^{(n-m-1)}(T(n-m)) \\ c_m(\sigma_0,\ldots,\sigma_{m-1},1-\sigma') \\ \iota \end{pmatrix}\right)\right] \\
&\quad+\sum_{m=0}^{n-1}\sum_{\sigma\in\{0,1\}^{m+1}}\sum_{\iota\in\mathscr{I}_{n-m-1,m+1}}\sum_{h'=2}^{1+2^{n-m-1}}(-1)^{\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_{h'}^{(n-m-1)}(T(n-m)) \\ c_{m-1}(\sigma_0,\ldots,\sigma_{m-1}),2^{-1} \\ \iota \end{pmatrix}\right)\right]
\end{aligned}$$

$$+\sum_{m=0}^{n-1}\sum_{\sigma\in\{0,1\}^{m+1}}\sum_{\iota\in\mathscr{I}_{n-m-1,m+1}}\sum_{h'=2}^{1+2^{n-m-1}}(-1)^{\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_{h'}^{(n-m-1)}(T(n-m)) \\ c_m(\sigma_0,\dots,\sigma_{m-1},1-\sigma_m) \\ \iota \end{pmatrix}\right)\right]$$

$$+\sum_{i'=2}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}(-1)^{1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} {}^{\mathrm{t}}(2^{-1},\dots,2^{-1}) \\ c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'}) \\ \iota \end{pmatrix}\right)\right]$$

$$+\sum_{i'=2}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}(-1)^{1+1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_1^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'}) \\ \iota \end{pmatrix}\right)\right]$$

$$+\sum_{i'=2}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}(-1)^{\mathrm{tr}(\sigma)}\mathscr{S}(\iota)2^{i'-1}\left[f\left(s\begin{pmatrix} {}^{\mathrm{t}}(2^{-1},\dots,2^{-1}) \\ c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'}) \\ \iota \end{pmatrix}\right)\right]$$

$$+\sum_{i'=2}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}\sum_{h'=2}^{1+2^{i'-1}}(-1)^{1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_{h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'}) \\ \iota \end{pmatrix}\right)\right]$$

$$+\sum_{i'=3}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}\sum_{h'=1}^{a_{i'-2}}(-1)^{i'+\mathrm{tr}(\sigma)+\chi_{i'-2}(h')}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} {}^{\mathrm{t}}(2^{-1},\dots,2^{-1}) \\ c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'}) \\ \iota \end{pmatrix}\right)\right]$$

$$+\sum_{i'=3}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+2}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}\sum_{h'=1}^{(i'-1)a_{i'-2}}(-1)^{i'+1+\mathrm{tr}(\sigma)+\chi_{i'-2}(h')}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_{1+2^{i'-1}+\sigma_{n-i'+1}(i'-1)a_{i'-2}+h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'}) \\ \iota \end{pmatrix}\right)\right]$$

$$=\sum_{m=0}^{n-1}\sum_{\sigma\in\{0,1\}^{m+1}}\sum_{\iota\in\mathscr{I}_{n-m-1,m+1}}(-1)^{1+1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_1^{(n-m-1)}(T(n-m)) \\ c_m(\sigma) \\ \iota \end{pmatrix}\right)\right]$$

$$+0$$

$$+\sum_{m=0}^{n-1}\sum_{\sigma\in\{0,1\}^{m+1}}\sum_{\iota\in\mathscr{I}_{n-m-1,m+1}}\sum_{h'=2}^{1+2^{n-m-1}}(-1)^{1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_{h'}^{(n-m-1)}(T(n-m)) \\ c_m(\sigma) \\ \iota \end{pmatrix}\right)\right]$$

$$+0$$

$$+\sum_{i'=2}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}(-1)^{1+1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_1^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma) \\ \iota \end{pmatrix}\right)\right]$$

$$+0$$

$$+\sum_{i'=2}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+1}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}\sum_{h'=2}^{1+2^{i'-1}}(-1)^{1+\mathrm{tr}(\sigma)}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_{h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma) \\ \iota \end{pmatrix}\right)\right]$$

$$+0$$

$$+\sum_{i'=3}^{n+1}\sum_{\sigma\in\{0,1\}^{n-i'+2}}\sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}}\sum_{h'=1}^{(i'-1)a_{i'-2}}(-1)^{i'+1+\mathrm{tr}(\sigma)+\chi_{i'-2}(h')}\mathscr{S}(\iota)\left[f\left(s\begin{pmatrix} a_{1+2^{i'-1}+\sigma_{n-i'+1}(i'-1)a_{i'-2}+h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma_0,\dots,\sigma_{n-i'}) \\ \iota \end{pmatrix}\right)\right]$$

$$
\begin{aligned}
= \quad & \sum_{m=0}^{n-1} \sum_{\sigma\in\{0,1\}^{m+1}} \sum_{\iota\in\mathscr{I}_{n-m-1,m+1}} (-1)^{1+1+\mathrm{tr}(\sigma)} \mathscr{S}(\iota) \left[f\left(s \begin{pmatrix} a_1^{(n-m-1)}(T(n-m)) \\ c_m(\sigma) \\ \iota \end{pmatrix}\right)\right] \\
& + \sum_{m=0}^{n-1} \sum_{\sigma\in\{0,1\}^{m+1}} \sum_{\iota\in\mathscr{I}_{n-m-1,m+1}} \sum_{h'=2}^{1+2^{n-m-1}} (-1)^{1+\mathrm{tr}(\sigma)} \mathscr{S}(\iota) \left[f\left(s \begin{pmatrix} a_{h'}^{(n-m-1)}(T(n-m)) \\ c_m(\sigma) \\ \iota \end{pmatrix}\right)\right] \\
& + \sum_{i'=2}^{n+1} \sum_{\sigma\in\{0,1\}^{n-i'+1}} \sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}} (-1)^{1+1+\mathrm{tr}(\sigma)} \mathscr{S}(\iota) \left[f\left(s \begin{pmatrix} a_{h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma) \\ \iota \end{pmatrix}\right)\right] \\
& + \sum_{i'=2}^{n+1} \sum_{\sigma\in\{0,1\}^{n-i'+1}} \sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}} \sum_{h'=2}^{1+2^{i'-1}} (-1)^{1+\mathrm{tr}(\sigma)} \mathscr{S}(\iota) \left[f\left(s \begin{pmatrix} a_{h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma) \\ \iota \end{pmatrix}\right)\right] \\
& + \sum_{i'=3}^{n+1} \sum_{\sigma\in\{0,1\}^{n-i'+1}} \sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}} \sum_{h'=1}^{2(i'-1)a_{i'-2}} (-1)^{1+\mathrm{tr}(\sigma)+\chi_{i'-1}(1+2^{i'-1}+h')} \mathscr{S}(\iota) \left[f\left(s \begin{pmatrix} a_{1+2^{i'-1}+h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma) \\ \iota \end{pmatrix}\right)\right] \\
= \quad & \sum_{\sigma\in\{0,1\}^{n}} \sum_{\iota\in\mathscr{I}_{0,n}} (-1)^{1+1+\mathrm{tr}(\sigma)} \mathscr{S}(\iota) \left[f \begin{pmatrix} c_{n-1}(\sigma)_{\iota(1)} \\ \vdots \\ c_{n-1}(\sigma)_{\iota(n)} \end{pmatrix}\right] \\
& + \sum_{\sigma\in\{0,1\}^{n}} \sum_{\iota\in\mathscr{I}_{0,n}} (-1)^{1+\mathrm{tr}(\sigma)} \mathscr{S}(\iota) \left[f \begin{pmatrix} c_{n-1}(\sigma)_{\iota(1)} \\ \vdots \\ c_{n-1}(\sigma)_{\iota(n)} \end{pmatrix}\right] \\
& +(-1)^{1+1}[f(a_{h'}^{(n)}(T(n+1)))] \\
& + \sum_{h'=2}^{1+2^{n}} (-1)^{1}[f(a_{h'}^{(n)}(T(n+1)))] \\
& + \sum_{i'=3}^{n} \sum_{\sigma\in\{0,1\}^{n-i'+1}} \sum_{\iota\in\mathscr{I}_{i'-1,n-i'+1}} \sum_{h'=1}^{2(i'-1)a_{i'-2}} (-1)^{1+\mathrm{tr}(\sigma)+\chi_{i'-1}(1+2^{i'-1}+h')} \mathscr{S}(\iota) \left[f\left(s \begin{pmatrix} a_{1+2^{i'-1}+h'}^{(i'-1)}(T(i')) \\ c_{n-i'}(\sigma) \\ \iota \end{pmatrix}\right)\right] \\
& + \sum_{h'=1}^{2na_{n-1}} (-1)^{1+\chi_{n}(1+2^{n}+h')}[f(a_{1+2^{n}+h'}^{(n)}(T(n+1)))] \\
= \quad & 0 \\
& +[f(t_1,\ldots,t_n))] \\
& + \sum_{I\in\{0,1\}^{n}} (-1)^{1} \left[f\left(\frac{t_1+I_1}{2},\ldots,\frac{t_n+I_n}{2} \right)\right] \\
& +0 \\
& + \sum_{\sigma_0\in\{0,1\}} \sum_{\iota\in\mathscr{I}_{n-1,1}} \sum_{h'=1}^{a_{n-1}} (-1)^{1+\sigma_0+\chi_{n-1}(h')} \mathscr{S}(\iota) \left[f\left(s \begin{pmatrix} a_{h'}^{(n-1)}(t_1,\ldots,t_n) \\ 1-\sigma_0 \\ \iota \end{pmatrix}\right)\right]
\end{aligned}
$$

$$= \quad (\mathrm{id} - B)[f] + \sum_{\sigma' \in \{0,1\}} \sum_{\iota \in \mathscr{I}_{n-1,1}} \sum_{h'=1}^{a_{n-1}} (-1)^{\sigma' + \chi_{n-1}(h')} \mathscr{S}(\iota) \left[f \left(s \begin{pmatrix} a_{h'}^{(n-1)}(t_1, \ldots, t_n) \\ \sigma' \\ \iota \end{pmatrix} \right) \right]$$

$$(\Phi_{n-1} \circ d_n) f = \sum_{h=1}^{a_{n-1}} (-1)^{\chi_{n-1}(h)} \left[(d_n f \circ a_h^{(n-1)}) \right]$$

$$= \quad \sum_{h=1}^{a_{n-1}} \sum_{i'=1}^{n+1} \sum_{\sigma'=0}^{1} (-1)^{i' + \sigma' + \chi_{n-1}(h)} \left[f \begin{pmatrix} a_h^{(n-1)}(t_1, \ldots, t_n)_1 \\ \vdots \\ a_h^{(n-1)}(t_1, \ldots, t_n)_{i'-1} \\ \sigma' \\ a_h^{(n-1)}(t_1, \ldots, t_n)'_i \\ \vdots \\ a_h^{(n-1)}(t_1, \ldots, t_n)_n \end{pmatrix} \right]$$

$$= \quad \sum_{h'=1}^{a_{n-1}} \sum_{\sigma'=0}^{1} \sum_{\iota \in \mathscr{I}_{n-1,1}} (-1)^{\sigma' + \chi_{n-1}(h)} \mathscr{S}(\iota) \left[f \left(s \begin{pmatrix} a_{h'}^{(n-1)}(t_1, \ldots, t_n) \\ \sigma' \\ \iota \end{pmatrix} \right) \right]$$

Thus one concludes

$$(d_{n+1} \circ \Phi_n - \Phi_{n-1} \circ d_n)[f] = (\mathrm{id} - B)[f].$$

□

6.2 Cohomological vanishing of a Stein space

We give a brief proof of the basic property of a non-Archimedean Stein space in the totally same way as that of a Stein space in the sense of a complex manifold in [GR].

Lemma 4.6.5. *Let X be a Stein k-analytic space, Definition 1.4.27. or a Stein k-dagger space, Definition 2.4.27, and F an arbitrary coherent sheaf on X. Then the cohomology group $H^i(X, F)$ vanishes for any $i \in \mathbb{N}_+$.*

Proof. Take a flasque resolution $(F^*, d^* \colon F^* \to F^{*+1})$ of F, and we prove the cochain complex $H^i(H^0(X, F^*))$ vanishes for any $i > 0$. For each sheaf F' on X and each analytic domain $V \subset X$ and , setting $F'_G(V) \equiv \varinjlim F'(U)$, where U in the limit runs through all open neighbourhood of V, extend F' to the sheaf F'_G on the G-topological soace X_G. A Stein space is a good analytic space in the sense of [BER2]. Since F is coherent, we calculate the cohomology group in the G-topology of X by the comparison theorem of coherent sheaves on a good analytic space ([BER2], 1.3.6/(ii)). Take a Weierstrass filtration $W_0 \subset W_1 \subset \cdots \subset \cup W_i = X$, and then the restriction map $F^i_G(W_{j+1}) \to F^i_G(W_j)$ is surjective for each $i \in \mathbb{N}$ and $j \in \mathbb{N}$, because $W_j \subset \mathrm{Int}(W_{j+1}/X)$ by the definition of a Weierstrass filtration. Furthermore, the restriction $F^i|_{W_j}$, which is the pull-back of F^i by the inclusion $W_j \to X$, is a flasque sheaf on the ordinary topology of W_j, and the cochain complex $(F^*|_{W_j}, d^*)$ is a flasque resolution of the coherent sheaf $F|_{W_j}$ for each $j \in \mathbb{N}$. Note

that the sheaf $(F|_{W_j})_G$ coinceds with $F_G|_{W_j}$ for each $j \in \mathbb{N}$ by the comparison theorem of coherent sheaves. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & d^{i-1}(F^{i-1}(X)) & \longrightarrow & F^i(X) & \longrightarrow & d^i(F^i(X)) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & d^{i-1}(F_G^{i-1}(X)) & \longrightarrow & F_G^i(X) & \longrightarrow & d^i(F_G^i(X)) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & d^{i-1}\left(\varprojlim_{j\in\mathbb{N}} F_G^{i-1}(W_j)\right) & \longrightarrow & \varprojlim_{j\in\mathbb{N}} F_G^i(W_j) & \longrightarrow & d^i\left(\varprojlim_{j\in\mathbb{N}} F_G^i(W_j)\right) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \varprojlim_{j\in\mathbb{N}} d^{i-1}(F_G^{i-1}(W_j)) & \longrightarrow & \varprojlim_{j\in\mathbb{N}} F_G^i(W_j) & \longrightarrow & \varprojlim_{j\in\mathbb{N}} d^i(F_G^i(W_j)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & d^{i-1}(F_G^{i-1}(W_j)) & \longrightarrow & F_G^i(W_j) & \longrightarrow & d^i(F_G^i(W_j)) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & d^{i-1}(F^{i-1}|_{W_j}(W_j)) & \longrightarrow & F^i|_{W_j}(W_j) & \longrightarrow & d^i(F^i|_{W_j}(W_j)) & \longrightarrow & 0
\end{array}$$

for each $i \in \mathbb{N}_+$ and $j \in \mathbb{N}$. What we want to verify is the exactness of the first row. If we have done, then we obtain $H^i(X, F) = 0$ for any $i > 0$. Since W_j is an affinoid space, the last row is exact by Kiehl's Theorem B (strict case: [KIE], 2.4.2. or [BGR], 9.5.3/3, and generalisation: [BER1], 2.1.11. and [BER2], 1.2.) and Tate's acyclicity theorem for affinoid algebras (strict case: [BGR], 8.2.1/1. and generalisation: [BER1], 2.2.5.) and hence so is the fifth row. Since the restriction map $F_G^i(W_{j+1}) \to F_G^i(W_j)$ is surjective, the system $F_G^i(W_*)$ satisfies the Mittag-Leffler condition for each $i \in \mathbb{N}$. Therefore the projective limit is exact, and thus the fourth row is exact for each $i > 0$. It follows that $\varinjlim d^i(F_G^i(W_j)) = d^i(\varinjlim F_G^i(W_j))$ for each $i > 0$ because the homomorphism $\varprojlim F_G^i(W_j) \to \varprojlim d^i(F_G^i(W_j))$ in the fourth row factors $\varprojlim F_G^i(W_j) = F_G^i(X) \to d^i(F_G^i(X)) = d^i(\varprojlim F_G^i(W_j))$, and hence the third row is exact for each $i \geq 2$. Therefore we have verified the exactness of the first row for each $i \geq 2$. In order to show the exactness of the first row for $i = 1$, we have to prove $\varinjlim d^0(F_G^0(W_j)) = d^0(\varinjlim F_G^0(W_j))$. Next,

consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_{j\in\mathbb{N}} F_G(W_j) & \longrightarrow & \varprojlim_{j\in\mathbb{N}} F^0_G(W_j) & \longrightarrow & d^0\left(\varprojlim_{j\in\mathbb{N}} F^0_G(W_j)\right) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & \varprojlim_{j\in\mathbb{N}} F_G(W_j) & \longrightarrow & \varprojlim_{j\in\mathbb{N}} F^0_G(W_j) & \longrightarrow & \varprojlim_{j\in\mathbb{N}} d^0(F^0_G(W_j)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_G(W_j) & \longrightarrow & F^0_G(W_j) & \longrightarrow & d^0(F^0_G(W_j)) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & F|_{W_j}(W_j) & \longrightarrow & F^0|_{W_j}(W_j) & \longrightarrow & d^0(F^0|_{W_j}(W_j)) & \longrightarrow & 0
\end{array}$$

for each $j \in \mathbb{N}$. By the same argument as above, it suffices to show the exactness at $\varprojlim d^0(F^0_G(W_j))$ in the second row in order to verify the equality $\varprojlim d^0(F^0_G(W_j)) = d^0(\varprojlim F^0_G(W_j))$. Since W_j is an affinoid space, the last row is exact by Kiehl's Theorem B and Tate's acyclicity theorem for affinoid algebras, and hence so is the third row. By the equivalence of the category of finite modules of an affinoid algebra and the category of finite Banach module of the affinoid algebra ([BER1], 2.1.9.), and by the Kiehl's Theorem B again, regard the group of sections $F_G(W_j)$ as an A_{W_j}-Banach module, and replacing their norms to an equivalent one in order from $F_G(W_0)$, we assume the restriction map $F_G(W_j) \to F_G(W_{j'})$ is a contraction map for each $j' \leq j \in \mathbb{N}$. Take an arbitrary element $f = (f_j)_{j\in\mathbb{N}} \in \varprojlim d^0(F^0_G(W_j))$. We want to construct a sequence $g = (g_j) \in \prod F^0_G(W_j)$ such that $d^0(g_j|_{W_{j'}}) = f_{j'}$ for each $j' \leq j \in \mathbb{N}$ and the element $g_{j+1}|_{W_j} - g_j \in (\ker d^0)(W_j)$ satisfies $\|(g_{j+1}|_{W_j} - g_j)\|_{F_G(W_j)} < 2^{-j}$ as an element of the A_{W_j}-Banach mogule $F_G(W_j) = (\ker d^0)(W_j)$ for each $j \in \mathbb{N}$. The homomorphism $d^0 \colon F^0_G(W_0) \to d^0(F^0_G(W_0))$ is surjective by the definition of the group-theoretical image $d^0(F^0_G(W_0))$, and there exists some section $g_0 \in F^0_G(W_0)$ such that $d^0 g_0 = f_0$. Suppose we have constructed a sequence $(g_0, \ldots, g_m) \in F^0_G(W_0) \times \cdots \times F^0_G(W_m)$ of finite length satisfying the desired condition above for an integer $m \in \mathbb{N}$. The homomorphism $d^{m+1} \colon F^0_G(W_{m+1}) \to d^{m+1}(F^0_G(W_{m+1}))$ is surjective by the definition of the group-theoretical image $d^{m+1}(F^0_G(W_{m+1}))$, and there exists some section $g'_{m+1} \in F^0_G(W_{m+1})$ such that $d^{m+1} g_{m+1} = f_{m+1}$. Set $\delta'_m \equiv g'_{m+1}|_{W_m} - g_m \in F^0_G(W_m)$. Since W_m is a Weierstrass domain of W_{m+1}, the image of the bounded k-homomorphism $A_{W_{m+1}} \to A_{W_m}$ is dense in A_{W_m}. Therefore considering the contractive admissible isomorphism $F_G(W_{m+1}) \hat{\otimes}_{A_{W_{m+1}}} = F_G(W_{m+1}) \otimes_{A_{W_{m+1}}} A_{W_m} \to F_G(W_m)$ as $A_{W_{m+1}}$-Banach modules, the image of the restriction map $F_G(W_{m+1}) \to F_G(W_m)$ is dense in $F_G(W_m)$. Consequently there exists some $\delta''_{m+1} \in F_G(W_{m+1})$ such that $\|\delta''_{m+1}|_{W_m} - \delta'_m\|_{F_G(W_m)} < 2^{-m}$. Set $g_{m+1} \equiv g'_{m+1} - \delta''_{m+1}$. Now we have $g_{m+1}|_{W_m} - g_m = (g'_{m+1} - \delta''_{m+1})|_{W_m} - g_m = \delta'_m - \delta''_{m+1}|_{W_m} \in F_G(W_m) = (\ker d^0)(W_m)$ and $\|\delta'_m - \delta''_{m+1}|_{W_m}\|_{F_G(W_m)} < 2^{-m}$. Thus we have constructed the required sequence $(g_0, \ldots, g_{m+1}) \in F^0_G(W_0) \times \cdots \times F^0_G(W_{m+1})$. By the axiom of dependent choice,

there exists such a sequence $g = (g_j) \in \prod F_G^0(W_j)$ satisfying the desired condition above. Set $\delta_j \equiv g_{j+1}|_{W_j} - g_j \in F_G(W_j)$ for each $j \in \mathbb{N}$. By the condition of g, we have $\|\delta_j|_{W_{j'}}\|_{F_G(W_{j'})} \leq \|\delta_j\|_{F_G(W_j)} < 2^{-j}$ for each $j' \leq j \in \mathbb{N}$, and hence the infinite sum $\sum_{l \leq j} \delta_l|_{W_j}$ converges in $F_G(W_j)$ for each $j \in \mathbb{N}$. Set

$$h = (h_j)_{j \in \mathbb{N}} \equiv \left(g_j + \sum_{l=j}^{\infty} \delta_l|_{W_j} \right)_{j \in \mathbb{N}} \in \prod_{j=0}^{\infty} F_G^0(W_j),$$

and then we have

$$\begin{aligned} h_{j+1}|_{W_j} &= \left. \left(g_{j+1} + \sum_{l=j+1}^{\infty} \delta_l|_{W_{j+1}} \right) \right|_{W_j} = g_{j+1}|_{W_j} + \sum_{l=j+1}^{\infty} \delta_l|_{W_j} \\ &= g_j + \delta_j + \sum_{l=j+1}^{\infty} \delta_l|_{W_j} = h_j \end{aligned}$$

by the continuity of the restriction map $F_G(W_{j+1}) \to F_G(W_j)$ for each $j \in \mathbb{N}$. It follows that $h \in \varprojlim F_G^0(W_j)$, and $d^0 h = d^0 g + (\sum_{l \leq j} (d^0 \delta_l)|_{W_j})_{j \in \mathbb{N}} = f + 0 = f$, which was what we wanted. □

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